Multi-level atoms and multimode fields

The atomic energies are $E_i$. Suppose we consider the atom in state 1 with $N_a$, $N_b$, & $N_c$ photons in the light field modes. The Bohr criteria suggests that a single photon can be removed from mode “a”, while the atoms is excited to energy level 2, approximately conserving energy. From there, adding a photon to mode “b” while lowering the atomic energy to $E_3$ also nearly conserves energy. Finally, the removal of a photon from “c” excites the atom to level 4.

Note that we are assuming that modes “b” and “c” can’t cause transitions between states 1 and 2. This is only a good approximation if all the frequencies $\omega_a$, $\omega_b$, $\omega_c$ are quite different.

Thus the set of atom+photon states that are nearly degenerate is:

$|1, N_a, N_b, N_c\rangle$, $|2, N_a - 1, N_b, N_c\rangle$, $|3, N_a - 1, N_b + 1, N_c\rangle$, $|4, N_a - 1, N_b + 1, N_c - 1\rangle$.

If we take the first state to have energy=0, the others have $E_{21} - \hbar \omega_a$, $E_{31} - \hbar \omega_a + \hbar \omega_b$, $E_{41} - \hbar \omega_a + \hbar \omega_b - \hbar \omega_c$, where for example

$E_{21} = \frac{\hbar \Omega^0}{2}$

Thus, the approximate Hamiltonian for this situation is

$$H = \begin{pmatrix}
0 & \frac{\hbar \Omega^0}{2} & 0 & 0 \\
\frac{\hbar \Omega^0}{2} & E_{21} - \hbar \omega_a & \frac{\hbar \Omega^0}{2} & 0 \\
0 & \frac{\hbar \Omega^0}{2} & E_{31} - \hbar \omega_a + \hbar \omega_b & \frac{\hbar \Omega^0}{2} \\
0 & 0 & \frac{\hbar \Omega^0}{2} & E_{41} - \hbar \omega_a + \hbar \omega_b - \hbar \omega_c
\end{pmatrix}$$

Each of the Rabi frequencies would be vacuum Rabi frequencies $\Omega^0$ multiplied by $\sqrt{N}$ factors. For example, $\hbar \Omega_b = 2 \langle 2, N_a - 1, N_b, N_c | e \sigma_z \sin(kz) a_0 | 3, N_a - 1, N_b + 1, N_c \rangle = \sqrt{N_b + 1} \Omega^0_b$. 