Heisenberg Picture

In the Schrödinger approach to quantum mechanics, operators \( \hat{O} \) are time-independent and the wavefunction of the system carries all the time-dependent information. The expectation value of \( \hat{O} \) obeys the Ehrenfest relation

\[
i \hbar \frac{d \langle \hat{O} \rangle}{dt} = \left[ \hat{O}, \hat{H} \right]
\]

Buried in this is the time-dependent wavefunction

\[
|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\psi(0)\rangle = U(t) |\psi(0)\rangle
\]

If we make this substitution into an expectation value we get

\[
\langle \psi(t) | O | \psi(t) \rangle = \langle \psi(0) | U^{-1}(t) O U(t) | \psi(0) \rangle \equiv \langle \psi(0) | O(t) | \psi(0) \rangle
\]

where in this Heisenberg picture the (time-independent) Schrödinger operator \( O \) becomes time-dependent:

\[
O(t) = U^{-1}(t) O U(t)
\]

The Heisenberg operator \( O(t) \) obeys an equation similar to the Ehrenfest equation, but without the expectation values:

\[
i \hbar \frac{d \hat{O}(t)}{dt} = \left[ \hat{O}(t), \hat{H} \right]
\]

Such an equation is understood to operate on a time-independent wavefunction \( |\psi_0\rangle = |\psi(0)\rangle \).

An example of the usefulness of this is to discover the operator corresponding to the vector potential \( A_x \) of a single mode of the electromagnetic field, given the electric field operator

\[
E_x(t) = \mathcal{E}(a(t) + a^\dagger(t)) \sin(k z) = -\frac{d A_x(t)}{dt}
\]

Comparing this to Eq. (1) gives

\[
\frac{d A_x(t)}{dt} = -\frac{i}{\hbar} [A_x(t), \hat{H}] = -\mathcal{E} (a(t) + a^\dagger(t)) \sin(k z)
\]

Since \([a, \hat{H}] = \hbar \omega a, [a^\dagger, \hat{H}] = -\hbar \omega a^\dagger\), we get \( \frac{d a(t)}{dt} = -i \omega a(t) \) so \( a(t) = a e^{-i\omega t}, a^\dagger(t) = a^\dagger e^{i\omega t} \) so

\[
E_x(t) = \mathcal{E} (a e^{-i\omega t} + a^\dagger e^{i\omega t}) \sin(k z) = -\frac{d A_x(t)}{dt}
\]

so

\[
A_x(t) = \frac{-i \mathcal{E}}{\omega} (a e^{-i\omega t} - a^\dagger e^{i\omega t}) \sin(k z)
\]

so the correct Schrödinger vector potential operator is

\[
A_x = A_x(0) = \frac{-i \mathcal{E}}{\omega} (a - a^\dagger) \sin(k z)
\]