1. Wave packets:

A quantum mechanical wave function $\Psi(x)$ has the same mathematical form as an electromagnetic wave $E(x)$. For instance, a plane wave is $\Psi(x,t) = \Psi_0 e^{i(kx-\omega t)}$. This has an infinite spatial extent.

As in an EM wave, producing a finite wave packet requires a range of $k$ values.

Assume a Gaussian envelope $e^{-\frac{x^2}{2\sigma_x^2}}$ modulates a plane wave with $k = k_0$ to make a finite length packet. Call the Fourier transform of this $\Psi(k)$, which is the required distribution of $k$ values.

In quantum mechanics, $|\Psi(x)|^2$ is the probability distribution for finding the particle at position $x$. Similarly $|\Psi(k)|^2$ is the probability distribution of $k$. The particle’s momentum $p$ is given by $p = \hbar k$. (So for $\Psi(x)$ a plane wave, it has an equal probability for being anywhere in space, but you know exactly what $k$ and $p$ are.)

Show how you would find the product $\Delta p \Delta x$, where $\Delta x$ is the r.m.s. variance* of $|\Psi(x)|^2$ and $\Delta p$ is the r.m.s. variance of $|\Psi(k)|^2$.

(A Gaussian profile turns out to be the shape with the smallest value of this product — you can try it with other function pairs, such as rectangle and sinc.)

*For the Gaussian distribution $e^{-\frac{t^2}{2\Delta^2}}$, $t$ has an r.m.s. variance $= \Delta$. 

$$
\Delta_{t,\text{r.m.s.}} = \sqrt{\int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2\Delta^2}\right) dt}
$$


In-class exercise — 15 March 2013

Group names: ____________________  ____________________  ____________________
2. Quadratic dispersion:

a) If $k = \frac{n\omega}{c}$, and $n = a + b\omega$, write the first three terms a Taylor expansion for $k$.

b) Suppose only the first two terms in the expansion of $k$ are non-zero, and that the coefficients are $s$ and $t$. Find $n(\omega)$. This is the form of $n$ that has a group velocity but no change in shape with propagation.

Note that in case a), even though $n$ is linear in $\omega$, $k$ is not. This means that the shape of a pulse will change as it propagates in this medium.

The change in shape can't be calculated in general, but the text did it for a Gaussian pulse for the case where there are no non-zero terms past the third.

Note that once the pulse leaves this material, if it is propagating in a vacuum, the shape will quit changing and whatever it looked like when it emerged will propagate with phase and group velocities = $c$. (If it is in a medium a $k$ of form b), it can have different group and phase velocities, but the shape will not change.)

While this can't be done in general, you can see what happens to a Gaussian pulse using 7.43 – 7.48.
3. Given \( e^{-i \frac{(t-z/v_g)^2}{2T^2(z)}} \Phi(z) + i(k_0 z - \omega_0 t) + i \frac{1}{2} \tan^{-1} \Phi(z) \)

where \( \Phi(z) \equiv \frac{2\alpha}{T^2} z \). If you put this in the form \( e^{-i(kz-\omega t)} \), what is the limit of \( \omega \) as \( z \to \infty \)?

(Remember this will be everything that is multiplied by \( t \)).
4. Suppose you have two identical gratings with line spacing $d$. You place their grooved faces parallel and facing each other 2 cm apart. A beam containing two plane waves of wavelength $\lambda_1$ and $\lambda_2$ travelling at an angle theta to the grating face normal is incident on one of the gratings, is diffracted to the other, and diffracted back so that it escapes between the gratings. Sketch what this looks like and find the path difference for the two wavelengths.

The grating equation that relates the incoming and outgoing angles is

$$m\lambda = 2d(\sin(\theta_i) + \sin(\theta_r)).$$