In-class exercise — 8 March 2013

Group names: ____________________ ____________________ ____________________

1. Two plane waves polarized in x travel in the z direction with velocity \( c = 100 \text{ m/s} \). One has frequency 100 Hz and peak amplitude 1.0 V/m, the other frequency 101 Hz and amplitude 1.01 V/m. The x field at any point in space and time is just the sum of the two individual fields.

a) Sketch the resulting wave (net x component of the electric field) at some instant in time — don’t worry about being quantitative.

b) What is the wavelength of the “beat” (distance between points of maximum amplitude)?

c) What is the frequency with which these peaks pass a fixed point?

d) What is the difference between the amplitude at this peak point and the minimum height half way between?

e) What is the difference in *intensity* between these points?
2. Generalize the above problem so the two waves have frequencies $\omega$ and $\omega+\Delta\omega$ and wavenumbers $k$ and $k+\Delta k$. Write down the sum and see if you can use trig identities or complex variable manipulation to get it into a form that looks like the product of a plane wave and a slowly varying amplitude. (This is quite easy in our standard exponential notation — just remember than adding in an exponent is the product of exponentials.)

3. The beam of light from a laser is a very pure sinusoid of angular frequency $\omega_0$ (on the order of $10^{15}$ s$^{-1}$ for optical frequencies). It passes through a very fast shutter that is open for only $10^{-10}$ s, so a 3 cm long chunk of the beam is all that passes.

a. Think of two simple functions that can be combined by multiplication or convolution that will reproduce this chunk. (In this case “simple” means ones where you already know their Fourier transforms.)

b. What does this tell you is the frequency content of the pulse? How did you get this? You can see that it is no longer just $\omega_0$.

c. Note that the power (i.e., the intensity of the light) at each frequency is obtained by squaring the magnitude of the light’s amplitude at that frequency, just as we got the intensity at any time by squaring the amplitude at that time. What do we mean by “at” a frequency when there is a continuous distribution of frequencies? Do you know the term “spectral density”?
4. Instead of shaping the pulse with sharp ends, we open the shutter gradually and “taper” the amplitude of the pulse so it is Gaussian in time with r.m.s width = $\sigma$.
Note that a Gaussian with mean square deviation $\sigma^2$ is given by $\exp((x-x_0)^2/(2\sigma^2))$ with suitable normalization.

a) again find two simple functions you can combine to get this pulse, and use their Fourier transforms to find the frequency spectrum of the resulting pulse.

b) If the amplitude of the pulse has a Gaussian profile, what is the shape of the \textbf{intensity} pulse? What is its r.m.s. width?
Basic Fourier Transforms

\[ \mathcal{F}\{g(t)\} = G(f) = \int_{-\infty}^{\infty} g(t) e^{-2\pi i ft} \, dt \]

Convolution Theorem:

\[ g \ast h(t) = \int_{-\infty}^{\infty} g(y) h(t-y) \, dy \]

Also:

\[ g \ast h \Rightarrow G \ast H \]

Useful Extras:

Similarity: \( g(at) \Rightarrow G(\frac{f}{a}) \) (if \( g(t) \Rightarrow G(f) \))

Shift: \( g(t-a) = G(f) e^{-2\pi i af} \)

\( g \text{ real} \Rightarrow G(-t) = G^*(t) \) : Both real \( \Rightarrow \) Both symmetrical

\[ \text{Parseval: } \int_{-\infty}^{\infty} g^* g \, dt = \int_{-\infty}^{\infty} G^* G \, df \]