Relativistic Kinematics

Recall: \( E^2 = c^2 p^2 + (m_0 c^2)^2 \)
\( \beta_p = \frac{cp}{E}, \quad \gamma_p = \frac{E}{m_0 c^2} = \frac{1}{\sqrt{1 - \beta^2}} \)

\((E, \vec{p})\) transform like \((ct, \vec{x})\)

So we transform along the \(z\)-axis with \(\vec{p}\) in the \(xz\)-plane:

\[
\begin{align*}
x' &= x \\
y' &= y \\
z' &= \gamma (ct - \beta_p z) \\
p_x' &= p_x \\
p_y' &= p_y \\
p_z' &= \gamma (cp \cos \theta - \beta p_z) \\
E' &= \gamma (E - \beta p_z) \\
E' &= \gamma (E - \beta cp \cos \theta)
\end{align*}
\]

Consider the ratio:

\[
\frac{cp' \sin \theta'}{cp' \cos \theta'} = \tan \theta' = \frac{cp \sin \theta}{\gamma (cp \cos \theta - \beta E)} = \frac{\sin \theta}{\gamma (\cos \theta - \beta \frac{p_z}{p}}
\]

where \(\beta_p = \frac{cp}{E}\)

Also recall: Invariants include:

\(E^2 - c^2 p^2 = \text{constant} = (m_0 c^2)^2 > 0\)

\((E, \vec{p})\) are vectors in a 4-dimensional non-Euclidean space.

Note: All 4-vectors transform the same way under a Lorentz transformation.

The mass \((m_0)\) is Lorentz invariant.

Zero mass particles have \(E = cp, \beta = 1 \Rightarrow \gamma_p\text{ not defined.}\)

They travel at speed \(c\) and have no rest frame.

Consider a two-body collision: \(m_1 + m_2 \rightarrow m_3 + m_4\)

Initially:

\[
\begin{align*}
p_1' &\rightarrow p_3' \\
p_2 &\rightarrow z\text{-axis} \\
p_3' &\rightarrow z\text{-axis}
\end{align*}
\]

Final state:
\[ \vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4 \]
\[ E_1 + E_2 = E_3 + E_4 \quad \Rightarrow \quad p_3 \sin \theta_3 = p_4 \sin \theta_4 \]
\[ p_3 \cos \theta_3 + p_4 \cos \theta_4 = p_1 - p_2 \]

In these collisions, the masses can change. Consider \((m_3 + m_4) > (m_1 + m_2)\) or \((m_3 + m_4) < (m_1 + m_2)\); then mass has been converted into kinetic energy. Energy is conserved, but not mass.

Let us consider conservation of energy and momentum in a two-body decay in the center of mass of the parent. So \(m_0: \text{initial mass} \rightarrow (m_1 + m_2)\) where \((m_1 + m_2) < m_0\).

\[ E = m_0 c^2 = E_1 + E_2 \]

**Parent rest frame:**

\[ \vec{p}_1 = -\vec{p}_2 \]

Solve for \(E_1\):
\[ (m_0 c^2 - E_2)^2 = E_1^2 = (m_2 c^2)^2 + c^2 p^2 \]
\[ = (m_0 c^2)^2 - 2E_1 m_0 c^2 + E_1^2 \]
\[ = (m_0 c^2)^2 - 2E_1 m_0 c^2 + (m_1 c^2)^2 + (c p)^2 \]
\[ \Rightarrow \quad E_1 = \frac{(m_0 c^2)^2 + (m_1 c^2)^2 - (m_2 c^2)^2}{2 m_0 c^2} \]

For \(E_2\) interchange 1 and 2 \(\Rightarrow\)
\[ E_2 = \frac{(m_0 c^2)^2 + (m_2 c^2)^2 - (m_1 c^2)^2}{2 m_0 c^2} \]

For momenta:
\[ c^2 p^2 = E_1^2 - (m_1 c^2)^2 \]
\[ = (m_0 c^2)^4 + (m_1 c^2)^4 + (m_2 c^2)^4 - 2(m_1 c^2) (m_2 c^2) - 2(m_1 c^2) (m_0 c^2) + 2(m_2 c^2) (m_0 c^2) \]
\[ = 4(m_0 c^2)^2 \]
Define: \( \chi (x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz + 2xz \)

\[ \Rightarrow \quad c_p = \left[ x (m_1 c^2)^2 (m_2 c^2)^2 (m_3 c^2)^2 \right]^{1/2} \]

\[ \frac{2 m c^2}{\gamma} \]

In the parent rest frame the daughter products are in opposite directions.

Example: \( K^0 \rightarrow \pi^+ \pi^- \)

\( m c^2 = 495 \text{ MeV} \)

\( m_1 c^2 = m_2 c^2 = 140 \text{ MeV} \)

For this example the two final state masses are the same, so:

\[ E_1 = E_2 = \frac{m c^2}{\gamma} = 247 \text{ MeV} \]

\( \gamma = 1.76 \)

\[ 0.8 \quad \gamma \]

\[ c_p = \left( E^2 - (m c^2)^2 \right)^{1/2} = 203 \text{ MeV} \]

\( E_K = 197 \text{ MeV} \)

Example: \( \pi^0 \rightarrow \gamma \gamma \) (two massless photons)

\( m_{\pi} c^2 = 135 \text{ MeV} \)

\( E_1 = E_2 = \frac{135}{2} \text{ MeV} \)

Total mass conversion

Example: \( \pi^+ \rightarrow \mu^+ \nu_\mu \)

\( m c^2 = 140 \text{ MeV} \)

\( m_1 c^2 = 106 \text{ MeV} \)

\( m_2 c^2 = 0 \)

Since the neutrino is virtually massless, the momentum = \( E_2 \)

\[ \Rightarrow \quad E_1 = \frac{(m c^2)^2 + (m_1 c^2)^2}{2 m c^2} = 110 \text{ MeV} \quad \Rightarrow \quad E_K = 4 \text{ MeV} \]