We begin Chapter 2 of *special* relativistic mechanics. What is the relation between energy, linear momentum, and rest mass?

\[
E^2 = (\frac{p^2}{2m})^2 + mc^2 \quad E^2 = (cp)^2 + (mc^2)^2
\]

**Units:** 
- 1eV = 1 electron-volt = 1.6x10^{-19} joule
- 1MeV = 10^{-6} eV = 1.6x10^{-13} joule
- charge of e = 1.6x10^{-19} coulomb

**Masses**
- electron = 0.511 MeV
- proton = 0.938 MeV

\[
E^2 = (cp)^2 + (mc^2)^2 \Rightarrow E = \pm \left[ (cp)^2 + (mc^2)^2 \right]^{1/2}
\]

**Choose** (+) sign for real particles.

**Particle velocity:** 
\[
\beta = \frac{v}{c} = \frac{cp}{E}
\]

**Lorentz factor:** 
\[
\gamma = \frac{E}{mc^2}
\]

\[
\Rightarrow E = \gamma mc^2
\]
\[
p = \beta \gamma mc^2
\]

So while \( \beta < 1 \) always, \( E \) and \( cp \) can increase without limit, because of \( \gamma \).

The relativistic energy formula reduces to classical physics for \( \beta \ll 1 \).

Energy and linear momentum together make up a 4-vector, \((E, cp)\), with \((E)\) acting as the "time component." \((E, cp)\) Lorentz transform just like \((ct, \vec{x})\). To see the details, consider a Lorentz transformation going along the \(z\)-axis:

\[
\begin{align*}
\vec{X} &= (x')\hat{x} + (y')\hat{y} + (z')\hat{z} \\
x &= R\sin\phi \cos\theta \\
y &= R\sin\phi \sin\theta \\
z &= R \cos\phi
\end{align*}
\]
$\rightarrow (x', y')$ Lorentz invariant

\[ z' = \gamma (z - \beta ct) \quad c^2 t'^2 - z'^2 = \text{invariant} \]
\[ ct' = \gamma (ct - \beta z) \]

For an energy-momentum 4-vector:

\[ P_x' = P_x \text{ transverse p components Lorentz invariant} \]
\[ P_y' = P_y \]
\[ P_z' = \gamma (P_z - \frac{\beta E}{c}) \]
\[ E' = \gamma (\frac{E}{c} - \beta P_z) \]

\[ \alpha, \text{in terms of } (\Theta, \phi): \]

\[ p' \sin \Theta' \cos \phi = p \sin \Theta \cos \phi \quad \phi \text{ is Lorentz invariant} \]
\[ p' \sin \Theta' \sin \phi = p \sin \Theta \sin \phi \quad \phi \text{ is not Lorentz invariant} \]
\[ cP' \cos \Theta' = \gamma (cP \cos \Theta - \beta E) \quad \Theta \text{ is not Lorentz invariant} \]
\[ E' = \gamma (E - \beta cP \cos \Theta) \]

Recall, though: transverse components of velocity are not Lorentz invariant:

\[ u_x' = \frac{u_x}{\gamma (1 - \beta u_z)} \]

To simplify, we set $\phi = 0 \rightarrow$ only components $u_x, u_z$

\[ \beta_p = \frac{\sqrt{u_x^2 + u_z^2}}{c} \quad \gamma_p = \frac{1}{\sqrt{1 - \beta_p^2}} \]
\[ \beta_p' = \sqrt{u_x'^2 + u_z^2} \quad \gamma_p' = \frac{1}{\sqrt{1 - \beta_p'^2}} \]

Also, although proving this is tedious:

\[ cP'_{x'} = cP_x \quad \text{and} \quad \gamma_p', u_x' = \gamma_p u_x \]
Consider \( \beta = \beta_0 \), the particle moves along the \( +z \) axis, \( \Rightarrow \)
\[
\gamma' = \gamma \left( \frac{c}{\gamma} - \beta_0 E \right)
\]
However: \( \beta_0 = \frac{c}{\gamma} \Rightarrow \gamma' = 0 \)
\[
\Rightarrow E' = \gamma \left( E - \beta_0 c \right) = \gamma (1 - \beta_0^2) E = \frac{E}{\gamma} = \frac{\gamma_0 m_0 c^2}{\gamma_0}
\]
\[
= \gamma_0 \sqrt{m_0 c^2}
\]
\( \Rightarrow \) In its rest frame, the particle energy = mass energy.

The energy of a moving particle is \( E = \gamma m_0 c^2 \). The rest mass is Lorentz invariant. In fact, in analogy to:
\[
\sqrt{x^2 + y^2 - z^2} = \sqrt{c^2 t^2 - x^2}
\]
we have:
\[
E^2 - c^2 \beta^2 = (m_0 c^2)^2
\]
\( \Rightarrow \) the preserved "length" of the energy-momentum 4-vector is its mass.

Note, though, that the energy \( (E) \) is not Lorentz invariant.

Another point: In a given Lorentz reference frame with no forces, energy and linear momentum are separately conserved in a reaction (such as a collision) \( \Rightarrow \)
\[
\begin{align*}
px' &= px \\
py' &= py \\
px' &= pz \\
E' &= E
\end{align*}
\]

The quadratic relation \( E^2 = c^2 \beta^2 + (m_0 c^2)^2 \) leads to two non-classic results:
(1) The rest mass of a particle can be zero, \( \Rightarrow E = c \beta_0 \), \( \beta = 1 \), \( \forall \) not defined. A particle with zero rest mass has no rest frame. Such particles exist in nature, e.g., the photon.
(2) If \( m_0 c^2 \neq 0 \), the particle has a rest frame. It can decay into other, lighter particles and kinetic energy.