Let us consider the idea of Lorentz space-time invariant. What does this mean? It means:

\[ c^2t^2 - x^2 = \text{constant} \]

Pictorially:

If \((OP)^2 > 0\), events at coordinate \(P\) w.r.t. \(O\) are \("time-like\) are time-ordered in any Lorentz reference frame, and \(P\) could be causally connected to \(O\).

I will go through Lorentz transformations in some detail because some of the text figure & discussion are wrong.

The lines \(c^2t^2 - x^2 = 0\) separate the \("time-like\" (shaded) and \("space-like\" regions. For example:

\((OQ)\) is \("space-like\" since \((OQ)^2 < 0\). For \("space-like\" region, there is a Lorentz transformation in which \((O)\) and \((Q)\) are simultaneous.

Consider this in more detail. I draw Lorentz transformations on the diagram:

\[ \begin{align*}
ct &= c't' \\
ctp &= c'p' \\
xp &= x'p'
\end{align*} \]

In the \((t', x')\) reference frame \((O)\) and \((P)\) occur at the same value \(\beta(x')\); that is, at \(x' = 0\).

Recall:

\[ x' = \beta(x - \beta ct) \]
\[ ct' = \gamma( ct - \beta x) \]
and $x' - \beta c t' = 0 \Rightarrow \beta = \frac{x'}{c t'}$ which means

a coordinate transformation exists. Also, from the figure:

$\beta = \tan(\alpha)$

Consider $(0x)$: It is possible that $c t'' = c t - \beta x a = 0$. In this picture $\beta = \frac{c t}{x a}$ looks upside down, but it is ok because $x a > c t a$ for a space-like distance. Notice, though:

Only one axis is drawn - not both - for the other Lorentz transformation, and there is no projection from one axis to the other.

Your text sometimes discusses “Euclidean space rotations.” What is this? Pictorially:

\[ x' = x \cos \theta + y \sin \theta \]
\[ y' = y \cos \theta - x \sin \theta \]
\[ \sin^2 \theta + \cos^2 \theta = 1 \] which guarantees that:
\[ x'^2 + y'^2 = x^2 + y^2 \]

These rotations preserve the radius vector, which on page 71 is the length $(0p)$. As you know, the locus of all points on the circle of radius $(0p)$ in the $(xy)$ reference frame represents all possible coordinate pairs $(x', y')$ for the point $(p)$ under rotation.

Here’s the point: Are Lorentz transformations Euclidean space rotations? The Lorentz transformations:

\[ x' = x - \beta c t \]
\[ c t' = \gamma (c t - \beta x) \]

look similar to a Euclidean rotation; It is linear in all coordinates and has coefficients $(\beta)$ and $(\beta x)$
That cross couple like (sinc) and (cos). However, two problems are obvious:

1. \( y > 1 \) and the signs of \( (x') \) and \( (ct') \) are the same in \( (x') \) and \( (ct') \):

2. The constancy of the speed of light. It is \( c^2 t^2 - x^2 \) that is the preserved quantity.

\[
c^2 t^2 - x^2 = \text{constant}
\]

gives a rectangular hyperbola tangent to the light cone. The locus of all points:

\[
c^2 t^2 - x^2 = c^2 p^2 - x_p^2 \quad \text{in frame} \quad (0)
\]

represents all possible transformations of the event at \( (p) \) to other Lorentz coordinate frames. Pictorially:

For a space-like time-like transformation take this hyperbola. For a space-like coordinate:

These curves are conic sections, but not circles. The pictures suggest that the trigonometric functions can be replaced by hyperbolic functions. So I define a quantity \( (X) \) such that:

\[
tanh(X) = \beta \quad \Rightarrow \quad \cosh(X) = 1, \quad \sinh(X) = \beta X
\]

Recall:

\[
tanh(X) = \frac{\sinh(X)}{\cosh(X)} \quad \sinh(X) = \frac{e^X - e^{-X}}{2}
\]

\[
\cosh(X) = \frac{e^X + e^{-X}}{2}
\]
Then: 
$|\tanh(\tau)| \leq 1$

$cosh \tau > 1$

Also, hyperbolic functions obey the identity:

$cosh^2(\tau) - sinh^2(\tau) = 1$

which is equivalent to: $\tau^2 - x^2 = c^2 t^2 - 1$

if $\tau' = (cosh(\tau))x - (sinh(\tau))ct$

$ct' = (cosh(\tau))ct - (sinh(\tau))x$

This is not quite right - notice the two minus signs. To fix this, make the time axis imaginary. Then:

Length $= x^2 + (ict)^2 = x^2 - ct^2$

This makes time-like lengths negative, but we can deal with this. However, it also makes the rotation axis $iX$

imaginary:

$\cos(iX) = \frac{e^x + e^{-x}}{2} = cosh(x)$

$\sin(iX) = \frac{e^x - e^{-x}}{2} = sinh(x)$

With this:

$x' = [\cos(iX)]x - [i(\sin(iX))]ct$

$ict' = [\cos(iX)]ict + [\sin(iX)]x$

Notice that, unlike the Euclidean rotations (p. 22) the sign flips.

So if we use real coordinates - we get hyperbolas instead of circles and hyperbolic instead of

trigonometric functions. Introducing complex numbers makes the formulas look ok, but you
cannot do projective geometry because the angles are imaginary.

Note!! What I call \( \alpha \) - the text calls \( \alpha \) - as in \( \beta = \tan \alpha \), not \( \tan \alpha \). Also, note that \( \tan \alpha \approx \tanh \alpha \) if \( \alpha \approx 1 \).

So parts of some of the diagrams in the text are OK, especially if you use Lorentz transformation algebra and not projective trigonometry to calculate lengths.

For example: Parts of Fig. 1-27 are OK, because 
\( \cos \alpha = \frac{3.8}{3.17} = 0.75 \) is OK.

However, the length of the blue time line is:

\[
ct' = \sqrt{(31.7)^2 + (23.8)^2} = 39.6 \text{ is supposed to equal 21 months (!) which it clearly does not.}
\]

Another example: Fig. 1-28 is simply wrong. To begin with, the angle in the figure is \( \arctan (\beta) = 32\degree \), not \( 38\degree \). However, if you allow the figure correctly, it does not work.

Fig. 1-28 redrawn: \( \tan (\frac{\pi}{2} - \alpha) = \cot \alpha = \frac{h}{l'} \),
\( \cos(\alpha) = \frac{l'}{h} \), \( \sin(\alpha) = \frac{h}{l} \), \( \cot(\alpha) = \frac{l}{l'} \)

Pictorially:

\[
\begin{align*}
\theta & \leftarrow l' \rightarrow \leftarrow \alpha \\
\end{align*}
\]

Put in numbers: \( l = 8.7 \) cm
\( l' = 2.8 \) cm
\( \Rightarrow \frac{l'}{l} = 0.32 \)
\( \alpha = 38\degree \), but \( \beta = \tan \alpha = 0.78 \) \( \frac{\pi}{2} - \alpha = 52\degree \)
\[
\frac{1}{\sqrt{1 - \beta^2}} = 1.6 \quad \Rightarrow \quad 1 = 0.625
\]

\[
\cos \alpha = \frac{l'}{l} = \frac{l'}{l + \frac{\sin \alpha}{\cot \alpha}}
\]

\[
\Rightarrow \quad \frac{l'}{l} = \cos \alpha - \sin \alpha \frac{\cos 2\alpha}{\cot \alpha} = \frac{0.24}{0.79} = 0.3
\]

This ratio of (0.3) agrees with the construction of the figure but is a factor of (2x) too small to be \(1/2\).

As far as I know, it is not possible to treat two Lorentz frames of reference as a Euclidean rotation and place the frames on the same graph and connect them via an Euclidean rotation.

Now I shift topics to the Doppler effect for light.

Pictorially:

\[\text{(0) moves to the right w.r.t. (0) in the usual way. From (0) light of frequency } (f_0) \text{ is emitted. In time } (ct') \text{ it emits } (f_0 t') \text{ waves, which is a Lorentz invarient number}\]

\[\Rightarrow \quad N = f_0 t' = f t\]

where \(f\) is the frequency in \(\text{(0)}\).

Since \(ct = \gamma (ct' + \beta x')\) and \(x' = ct'\) for a light wave

\[\Rightarrow \quad ct = \gamma (1 + \beta) (ct') = \frac{1 + \beta}{\sqrt{1 - \beta^2}} (ct')\]

\[\Rightarrow \quad f t = \left(\frac{1 + \beta}{1 - \beta}\right)^{1/2} f t' = f_0 t'\]
\[ f = \left( \frac{1 + \beta}{1 - \beta} \right)^{1/2} f_0 \]

This means \( f \) is red-shifted (lower frequency) when \( (\theta') \) is receding and blue-shifted (higher frequency) when \( (\theta') \) is approaching \( (\theta) \).

In astronomy they define the red shift as:

\[ z = \frac{f_0 - f}{f} \]

\[ \Rightarrow \beta = \frac{(z+1)^2 - 1}{(z+1)^2 + 1} \]

Since galaxies with \( z \approx 5 \) have been found, such galaxies are receding at:

\[ \beta \approx 0.95. \]

To finish up chapter one:

* The description in the text of the twin paradox and the long pole in a short barn are OK, so are the associated diagrams, which do not show two Lorentz reference frames in the same plot.

* The world lines of the travelling twin are OK - but do not try to obtain time dilation by projection.

* Regarding the twins example: To distinguish which twin is younger you must go beyond special relativity because one twin accelerates while the other does not.