Physics 241

Lecture 10 Dec 2010

A few points about the hydrogen wave function.

\[ \psi_{n, l, m, e}(r, \theta, \phi) = \frac{R_{n l}(r)}{\sqrt{\langle \psi \rangle}} Y_{m l}(\theta, \phi) \]

Principal quantum number.

Spherical harmonics.

\[ \begin{align*}
\Psi_{200} &= \frac{2}{(2a_0)^{3/2}} (1 - \frac{n}{2a_0}) e^{-r/2a_0} Y_0^0 \\
\Psi_{210} &= \frac{1}{(2a_0)^{1/2}} e^{-r/2a_0} \sqrt{2} Y_1^1 \\
\Psi_{21-1} &= \frac{1}{(2a_0)^{1/2}} e^{-r/2a_0} \sqrt{2} Y_1^{-1} \\
\Psi_{21-1} &= -\left(\frac{3}{\sqrt{2}}\right) \sin \theta e^{-r/2a_0} Y_0^0 \\
\Psi_{211} &= \left(\frac{3}{\sqrt{2}}\right) \cos \theta e^{-r/2a_0} Y_0^0 \\
\Psi_{212} &= \left(\frac{3}{\sqrt{2}}\right) \sin \theta e^{-r/2a_0} e^{-i\phi} Y_0^0
\end{align*} \]

The radial and angular wave functions are separately normalized:

\[ \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_0^\infty r^2 dr |R_{n l}(r)|^2 = 1 \]

Note that \( R_{n l}(r) \to r^l \) as \( n \to 0 \), so only \( l=0 \) is finite at the origin. As \( n \) increases, \( l=0, 1, 2, \ldots, n-1 \), \( l=0 \) has the most zeros, because it has the highest degree polynomial; \( l=n-1 \) has no zeros, except at the origin.
$Y^m_n(\theta, \phi) \sim I^m_n(\cos \theta)e^{im\phi}$ is complex. Remember that I said that a bound state wave function could be chosen real, and that real wave functions have no current $\mathbf{j} = \frac{e}{2m} \left( \mathbf{r} \times \overrightarrow{\mathbf{v}} - (\mathbf{r} \cdot \mathbf{v}) \mathbf{v} \right)$. Well, if $m_2 \neq 0$ $\frac{\partial Y}{\partial \phi} = im \mathbf{Y}$, and the azimuthal component of $\mathbf{j}$ does not vanish, representing the sense of rotation of the electron current around the $z$ axis. This corresponds to the vector model of a fixed $\mathbf{J}$ precessing around the $z$ axis.

$|Y_{11}|^2 = \frac{3}{8\pi}$ gives,

combined with the radial distribution, sort of a bagel going around the $z$ axis, representing the $j$ plane perpendicular to $z$.

As $l$, $m_2$ increase, the bagel gets flatter. 'Circular orbits' have $l=n-1$ (max $l$ for a given energy) magnetic moments and the Zeeman effect.

$\mathbf{J} = \sum m \mathbf{A}$

$i = \frac{g}{4}$

$t = \text{period} = \frac{2\pi n}{v}$, $A = \pi n^2$

$\mathbf{p} = e \times \mathbf{A} \times \pi n^2 v \mathbf{v} = \frac{e}{2m} \mathbf{v} \mathbf{v} = \frac{e}{2m} \mathbf{r} \mathbf{v} = \frac{e}{2m} \mathbf{r} \mathbf{v} = \frac{e}{2m} \mathbf{L}$
The orbital magnetic moment \( \vec{\mu} \) has an energy

\[ U = -\vec{\mu} \cdot \vec{B} = -\frac{e}{2m} \vec{L} \cdot \vec{B} = -\frac{eB}{2m} L_z \text{ for } \vec{B} = \vec{B}_0 \]

Choose the \( z \)-axis - the axis of quantization, for the magnetic field direction. Since the eigenvalues of \( L_z \) are time, we have

\[ U = -\frac{e}{2m} B L_z \]

The quantity \( \mu_B = \frac{e}{2m} \) is called the Bohr magneton, where \( m \) is the electron mass.

\[ \mu_B = \frac{e}{2m} = 5.79 \times 10^{-5} \text{ eV/T} \]

\( B = 1 \text{ T} (10^4 \text{ gauss}) \) is a reasonable laboratory field.

For \( l = 1 \), \( m_L = +1, 0, -1 \)

\[
\begin{array}{c}
2p \\
0 \\
-1
\end{array}
\]

Since \( 2p \rightarrow 1s \) is 10.2 eV, this splitting is a tiny part of the wave length of the light, but similar splittings can be observed with a spectograph.
Electron spin \( \frac{1}{2}, \frac{3}{2} \) \( s = \frac{\hbar}{2} \sqrt{5(5+1)} = \frac{3}{4} \hbar \) \( S = \frac{1}{2} \)

\( S_z = \hbar m_s \) \( m_s = \pm \frac{1}{2} \) no zero! \( \frac{3}{4} \hbar \)

There is no space wave function which is an eigenfunction of \( S_z \) or \( S_z \). The spin wave functions are two non-parallel vectors \( |+\rangle \)

and \( |\rangle \) parity fixing \( S^+ |+\rangle = \frac{\hbar}{2} |\rangle \)

\( S^- |\rangle = -\frac{\hbar}{2} |+\rangle \)

\( S^+ |\rangle = \frac{3}{4} \hbar^2 |\rangle \), \( S^- |\rangle = \frac{3}{4} \hbar^2 |\rangle \)

\( \langle + | + \rangle = 1, \langle + | \rangle = 0 \), or the general normalization.

The spin operators \( S_x, S_y, S_z \) are \( 2 \times 2 \) matrices.

The electron has an intrinsic magnetic moment.

\[ \vec{p}_s = g \frac{e}{2m} \]

where \( g \), the Landé \( g \) factor, is defined by this equation.

For the electron, \( g = 2 \) almost exactly. Well,

\[ g_{1/2} - 1 = \frac{\alpha}{2\pi} = 0.00116 \]

\( \alpha = \frac{e^2}{\hbar c} \) being the fine structure constant, and \( g_{1/2} - 1 \) is called the 'anomaly.' It is small, but very important.
In an external magnetic field $U = -\mathbf{\mu} \cdot \mathbf{B}$

$U = -\frac{gZeS \cdot \mathbf{B}}{2m} = -gZeB \frac{S}{2m}$ for $m_S = \pm \frac{1}{2}$.

So if $g = 2$, $U = \pm \frac{e\hbar}{2m} B = \pm \mu_B B$

looks just like the orbital term, except there is no zero.

$$\frac{1}{2m} \mu_B B = (1.6 \times 10^{-5} \text{ eV/T}) \times B$$

Classical precession in a magnetic field: cyclotron and Larmor frequencies.

\[ \dot{\mathbf{B}} \text{ into page negative charge for } e^- \]

So $e^-$ goes clockwise.

\[ \frac{d\mathbf{P}}{dt} = e \mathbf{V} \times \mathbf{B} \quad \omega_L \mathbf{S} = e \mathbf{P} \times \mathbf{B} \quad \omega_c = \frac{eB}{m} \]

Cyclotron frequency.

Spin precession \[ \frac{d\mathbf{S}}{dt} = \mathbf{\mu} \times \mathbf{B} \]

\[ \omega_L \mathbf{S} = g \frac{eB}{2m} \]

or \[ \omega_L = \frac{eB}{m} = \omega_c \quad \text{if } g = 2. \] This equality holds relativistically, so if $g = 2$, the spin and the momentum precess together.
However, if $g \neq 2$, $g \frac{1}{2} - 1 = a$, the anomaly, then the spin and momentum get cut off at step, and the amount depends on $\frac{\Delta E}{mc^2}$.

$$\Delta \Theta = x a \frac{\Theta_{\text{bend}}}{2}$$

$\Delta \Theta$ is the angle through which the momentum $p^*$ bends. If $\Theta_{\text{bend}} = \frac{\pi}{2}$, $\Delta \Theta = \frac{x a \pi}{2}$.

Stanford Linear Collider: 50 GeV electrons

$$\gamma = \frac{E}{mc^2} \approx 10^5$$

The electron spin precesses $25 \times (2\pi)$ going around the bend at the SLAC. The electrons are 80% polarized, thanks to a source built by Prof. Perkins.

Addition of angular momenta $L^2 + \hat{S}^2$ in hydrogen:

$$\vec{J} = \vec{L}^2 + \vec{\hat{S}}^2$$

$$J^2 = \hbar^2 (j(j+1))$$

$J_z = \hbar m_j$

Given $l, s, m_l, m_s$, $m_j = m_l + m_s$ (they add)

$l + s \leq j \leq l + s$ in integer steps. If $s < l$
which is usually the case for hydrogen, then there are
\[ 2j+1 = 2 \text{ values of } j \]
\[ \frac{(2j+1)(2j+1)}{2} \]
number
of states
\[ 2j+1 \]
The states 
\[ (l, s, m_l, m_s) \]
are 
mixed to form 
\[ (l, j, m_j) \]

Note that for half integer \( j \), \( 2j+1 \) is even, so there
is no \( j = 0 \), while for integer \( j \) is not possible in
hydrogen, but possible in complex atoms 
(like for instance) then \( 2j+1 \) is odd, \( j = 0 \) exists.

Hydrogen

\[
\begin{array}{cccc}
m=1 & 0 & 1/2 & 1/2 & 1S1/2 \\
m=2 & 0 & 1/2 & 1/2 & 2S1/2 \\
m=2 & 1 & 1/2 & 1/2 & 2p1/2 \\
m=2 & 1 & 1/2 & 3/2 & 2p3/2 \\
m=3 & 0 & 1/2 & 1/2 & 3S1/2 \\
m=3 & 1 & 1/2 & 3/2 & 3p1/2, 3p3/2 \\
m=3 & 2 & 1/2 & 3/2, 5/2 & 3d3/2, 3d5/2 \\
\end{array}
\]

Vector addition
\[ \mathbf{J} = \mathbf{L} + \mathbf{S} \]
\[ m_l \]

\[ \mathbf{L} + \mathbf{S} \text{ process around } \mathbf{J} \]
since \( \mathbf{L} + \mathbf{S} \cdot \mathbf{J} \) are not constant, but \( \mathbf{J} \) is constant.
\[ J^2 = L^2 + S^2 + 2L \cdot S \]

or \[ L \cdot S = \frac{\hbar^2}{2} \left( J(J+1) - \ell(\ell+1) - \frac{3}{4} \right) \]

which is important in spin-orbit coupling.

The magnetic moment and Lande g factor is

\[ \vec{\mu}_J = \vec{\mu}_L + \vec{\mu}_S = \frac{e}{2m} \left( g_L \vec{L} + g_S \vec{S} \right) \]

\[ = \frac{e}{2m} \left( \vec{L}^2 + 2\vec{S} \right) = \frac{e}{2m} \left( \vec{J} + \vec{S} \right) \]

\[ \vec{\mu}_J \cdot \vec{B} = \frac{e}{2m} \left( \vec{J} \cdot \vec{B} + \vec{S} \cdot \vec{B} \right) = \frac{e}{2m} \left( \vec{J} \cdot \vec{B} + \vec{S} \cdot \vec{J} \right) \]

\[ = \frac{e}{2m} \left( \vec{J} \cdot \vec{B} + \frac{\vec{S} \cdot \vec{J}}{J^2} \cdot \vec{J} \cdot \vec{B} \right) = \frac{e}{2m} \left( 1 + \frac{\vec{S} \cdot \vec{J}}{J^2} \right) \frac{\vec{J} \cdot \vec{B}}{J^2} \]

The Lande g factor is

\[ g_L J = 1 + \frac{\vec{S} \cdot \vec{J}}{J^2} = 1 + \frac{S^2 - L^2}{2J^2} \]

\[ g_L J = 1 + \frac{\ell(\ell+1) + S(S+1) - L(L+1)}{2J(J+1)} \]

This formula is quantum mechanically correct.