Physics 241
Lecture 24 Nov 2010

Alpha decay

Virtually all heavy elements emit α-particles. The transition is \((Z, A) \rightarrow (Z-2, A-4) + \alpha\)

where \(\alpha\) has \(Z=2, A=4\) is a helium nucleus.

The \(\alpha\) energies are all in the \(\sim\) few MeV range, the lowest being perhaps 2MeV and the highest around 8 MeV.

**Examples from the back of the text** (eV)

\[
\begin{align*}
\text{Hf 174} & \rightarrow \text{Yb 170} & 2 \times 10^{15} \text{ years} & 2.47 \text{E}6 \\
\text{Os 186} & \rightarrow \text{W 182} & 2 \times 10^{15} \text{ years} & 2.82 \text{E}6 \\
\text{Pt 190} & \rightarrow \text{Os 186} & 6.5 \times 10^{11} \text{ years} & 3.2 \text{E}6 \\
\text{Th 228} & \rightarrow \text{Ra 224} & 1.9 \text{ years} & 5.5 \text{E}6 \\
\text{U 238} & \rightarrow \text{Th 234} & 4.5 \times 10^9 \text{ years} & 4.3 \text{E}6 \\
\text{Pb 210} & \rightarrow \text{Pb 206} & 138 \text{ days} & 5.4 \text{E}6 \\
\text{Am 241} & \rightarrow \text{Np 237} & 432 \text{ years} & 5.6 \text{E}6 \\
\text{Po 211} & \rightarrow \text{Pb 207} & 1/2 \text{ sec} & 7.6 \text{E}6 \\
\end{align*}
\]

So between Hf 174 \& Po 211 we have a variation in lifetime of over 23 orders of magnitude, with \(E\alpha\) increasing about a factor of 3.

Nuclear radii for α emission are about

\[
r_0 = 1.4 \times 10^{-15} \text{mm} \times (A)^{1/3}
\]

So for lead \(Z=82, A=208\)

\[
r_0 \approx 8.3 \times 10^{-6} \text{mm}
\]
The \( \alpha \) particle energy required to reach this radius is

\[
E_x = \frac{Z^2 e^2}{1} = \frac{2^2 \times 1\text{ keV}}{2^2 \times 1.44 \times 10^6} = 8.3 \text{ MeV}
\]

which is much greater than typical \( \alpha \) energies.

The \( \alpha \) particles must be emitted far below the Coulomb barrier. They tunnel out.

The \( \alpha \) particle is not bound inside the nucleus. It is only 'quasi-bound.' It penetrates the Coulomb barrier to get out.

The barrier thickness

\[
V = V_0 - V_b
\]

where \( \frac{Z^2 e^2}{4\pi\varepsilon_0 V_b} = E_x \)

\( Z = 2 \)

\( Z' = \) atomic number of the daughter nucleus. The \( \alpha \) is swimming around inside, unbound, but trapped by the Coulomb barrier.
Space-dependent wave number inside the barrier
\[ \frac{d^2 \psi}{dx^2} + k^2 \psi(x) = 0 \quad \text{where} \quad k^2 = \frac{2m}{\hbar^2} (V(x) - E) \]

If \( k \) is a constant, \( \frac{d^2 \psi}{dx^2} + k^2 \psi = 0 \) has solutions \( e^{\pm ikx} \). What happens if \( k \) is a function of \( x \)? The first-order solution to this problem is called the WKB approximation.

\[ \psi(x) \sim \frac{1}{\sqrt{k(x)}} e^{i \int k(x) dx} \]

The integral \( \int k(x) dx \) means the phase \( \Phi(x) \) depends less on \( x \) when \( k(x) \) is moving faster.

So \( k(x) \) is replaced by \( \int k(x) dx \) - the integral.

For our case \( k(x) = \left( \frac{2m}{\hbar^2} \right) \left( \frac{2Ze^2}{4\pi\varepsilon_0 r} - E \right) \)

\[ = \left( \frac{2mE}{\hbar^2} \right)^{\frac{1}{2}} \left( \frac{2Ze^2}{4\pi\varepsilon_0 r} \right)^{\frac{1}{2}} \left( \frac{1}{r} - 1 \right) \]

but \( r = \frac{2Ze^2}{4\pi\varepsilon_0 E} \)

So \( k(r) = \left( \frac{2mE}{\hbar^2} \right)^{\frac{1}{2}} \left( \frac{r}{r} - 1 \right)^{\frac{1}{2}} \)

and the transmission coefficient is proportional to \( e^{-\int k(x) dx} \).
The exponential is
\[ -\int \frac{d\mu}{\theta^2} = \left( \frac{2mE_0}{\hbar^2} \right)^{\frac{1}{2}} \int_0^{\frac{\pi}{2}} (\sec^2 \theta) d\theta \]

The integral looks hopeless, but it isn't.\]

Let
\[ \cos^2 \theta = \frac{\theta}{n} \]

\[ d\mu = 2 \sin \theta \cos \theta \ d\theta \]

\[ -\int \frac{d\mu}{\theta^2} = -\left( \frac{2mE_0}{\hbar^2} \right)^{\frac{1}{2}} \int_0^{\frac{\pi}{2}} \cos^{-1} \left( \frac{\theta}{n} \right) d\theta \]

\[ = -\left( \frac{2mE_0}{\hbar^2} \right)^{\frac{1}{2}} \left( \frac{n}{2} \sin^{-1} \left( \cos^{-1} \left( \frac{\theta}{n} \right) \right) \right) \]

\[ \Theta_0 = \cos^{-1} \left( \frac{\theta_0}{n} \right) \]

\[ \nu = \frac{2\pi e^2}{\hbar m E_0} \]

\[ -\int \frac{d\mu}{\theta^2} = -\frac{2\pi e^2}{\hbar m E_0} \left( \frac{2m^2}{\hbar^2} \right)^{\frac{1}{2}} \left( \Theta_0 - \frac{1}{2} \sin^{-1} \left( \cos \left( \frac{\theta_0}{n} \right) \right) \right) \]

For the transmission probability, double it:

\[ -2\int \frac{d\mu}{\theta^2} = -\frac{4\pi^2}{1.44} \left( \frac{2 \times 3.72 \times 10^8}{4 \times 10^4} \right)^{\frac{1}{2}} \left( \Theta_0 - \frac{1}{2} \sin^{-1} \Theta_0 \right) \]

\[ = -\frac{4\pi^2}{1.44} \left( \Theta_0 - \frac{1}{2} \sin^{-1} \Theta_0 \right) \]

\[ \Theta_0 = \cos^{-1} \left( \frac{\theta_0}{n} \right) \]
This is called Gamow's formula. The transition rate, which is the reciprocal of the lifetime, is 

\[ \frac{1}{\tau} \sim n \cdot \text{frequency for hitting the wall} \times e^{-2 \int_0^\infty x(n) \, dn} \]

So 

\[ \log_2 \tau = -\log_2 (\text{freq for hitting the wall}) \]

\[ + \log_2 2 \int_\infty^0 x(n) \, dn \]

If the first term, whatever it is, is sensibly constant, then \( \log_2 \tau \) should vary like 

\[ \frac{421 \times 621}{\sqrt{E_d}} \left( \Theta_0 - \frac{1}{2} \sin 2\Theta_0 \right) \quad \Theta_0 = \Theta_0^{10^{-101}} \]

This is pretty close to true. There are some outliers (like P211), but most nuclei follow this pattern fairly well.

So the model has several interesting features.

1. The \( x \) is unbound inside the nucleus, \( E_d > 0 \).
2. It tunnels through the Coulomb barrier to emerge as a free particle with \( E_d + KE \) at \( \infty \).
3. The nuclear well is approximated by a square well for lack of any better knowledge.
4. The tunnel thickness is \( 1 - \frac{1}{2} \), and the height at the nucleus is \( \frac{2E_d}{m \omega^2} \).
5. The same thing works in reverse: you can tunnel in!
α decay log lifetimes vs $E^{-5}$