Lecture 17: de Broglie

de Broglie hypothesis:

For photons: \( E = cp = h\nu = \frac{hc}{\lambda} \Rightarrow p = \frac{h}{\lambda} \)

His idea: this is true for rest mass particles as well. For non-relativistic physics:

\[ E_{\text{Kinetic}} = E_k = \frac{p^2}{2m} \Rightarrow p = \sqrt{2mE_k} \]

\[ \Rightarrow cp = \sqrt{2mc^2E_k} \Rightarrow \lambda = \frac{hc}{\sqrt{2mc^2E_k}} \]

= (for electrons, \( E_k \text{ in eV} \)) \( \frac{1.23 \text{ nm}}{\sqrt{E_k}} \)

Because of the electron mass, an electron wavelength is much shorter than that of a photon of the same kinetic energy.

For relativistic particles:

\[ cp = \sqrt{E^2 - (mc^2)^2} = \sqrt{E_k^2 + 2E_kmc^2} \]

\( (E_k = E - mc^2) \)

Let us look at Compton wavelength again:

\[ \lambda_{\text{Compton}} = \lambda_c = \frac{h}{mc} = \frac{hc}{mc^2} \]

If \( \lambda = \lambda_c \Rightarrow cp = mc^2 \Rightarrow \beta y = 1 \Rightarrow \beta = \sqrt{\frac{1}{y^2}} \sim 1 \text{ is} \text{ fairly relativistic} \)

Ratio: \[ \frac{\lambda}{\lambda_c} \approx \frac{mc^2}{cp} \]
Now consider slow neutrons:
\[ E_k \ll mc^2 = 937 \text{MeV} \]
\[ \Rightarrow \lambda = \frac{0.0286}{\sqrt{E_k}} \text{ (in nm) for neutrons} \]

To have \( \lambda \approx 0.1 \text{nm} \), \( E_k \) must be very small. At room temperature, \( k_B T = \frac{1}{40} \) eV
\[ \Rightarrow \lambda = 0.18 \text{nm} \]

\( \Rightarrow \) Thermal neutrons behave like 10keV x-rays or 100eV electrons. All are used to study crystal structure. Comparatively:

<table>
<thead>
<tr>
<th>&quot;wave&quot;</th>
<th>E</th>
<th>Advantages</th>
<th>Disadvantages</th>
</tr>
</thead>
<tbody>
<tr>
<td>X-ray</td>
<td>10keV</td>
<td>Penetrating</td>
<td>Destructive / hard to focus</td>
</tr>
<tr>
<td>e^-</td>
<td>100eV</td>
<td>Can be focused</td>
<td>only thin samples</td>
</tr>
<tr>
<td>n</td>
<td>( \frac{1}{40} ) eV</td>
<td>Penetrating + magnetic</td>
<td>Need lots of sample + difficult to handle + detect</td>
</tr>
</tbody>
</table>

Davison & Germer
1st measurement of e-diffraction

\[ \tan \alpha = \frac{h}{d} \Rightarrow \alpha = 36^\circ \]
\[ \phi = 2 \alpha = 52^\circ \]
\[ \lambda = 2d \sin \theta \]
\[ \theta + \alpha = \frac{\pi}{2} \]
\[ \sin \theta = \cos \left( \frac{\phi}{2} \right) \]
\[ d = D \sin (\alpha) = D \sin (\phi/2) \]
\[ \alpha = 2d \sin \theta = 2D \sin \left( \frac{\phi_2}{2} \right) \cos \left( \frac{\phi_2}{2} \right) = D \sin \phi \]

For Davison & Germer, who studied nickel:

\[ D = 0.21 \text{ nm} \]

\[ \Rightarrow \phi = 50^\circ, \quad E_k = 54 \text{ eV} \quad \text{(text)} \]

\[ \Rightarrow \lambda = \frac{1.23}{\sqrt{54}} = 0.167 \text{ nm} \]

while \((0.21) \sin 50^\circ = 0.161 \text{ nm}\)

Actually \((52^\circ)\) is a better fit to their data.

Now let us consider phase and group velocity:

\[ \lambda = \frac{h}{p} \]

\[ E = h \nu \]

Phase of a wave is written \((kx - \omega t)\), from

\[ \psi(x) = A e^{i(kx - \omega t)} \]

\[ \frac{dx}{dt} = \frac{\omega}{k} = \text{phase velocity} \]

\[ \omega = 2\pi \nu \quad k = \frac{2\pi}{\lambda} \]

\[ p = \hbar k \]

\[ E = \hbar \nu \]

For matter waves: \[ E = \frac{1}{2}mv^2 \]

\[ p = mv \]

\[ \Rightarrow \frac{\omega}{k} = \frac{E}{p} = \frac{\nu}{\lambda}. \quad \text{Should be } \nu \]

What to do?
Group velocity - beats.

Consider two waves of equal amplitude and slightly different frequency

\[ \psi(x,t) = A e^{i(k_1 x - w_1 t)} + e^{i(k_2 x - w_2 t)} \]

Let \[ k = \frac{k_1 + k_2}{2}, \quad w = \frac{w_1 + w_2}{2}, \quad \Delta k = \frac{k_1 - k_2}{2}, \quad \Delta w = \frac{w_1 - w_2}{2} \]

=> \[ \psi(x,t) = A e^{i[(k + i\Delta k)x - (w + i\Delta w)t]} + e^{i[(k - i\Delta k)x - (w - i\Delta w)t]} \]

=> \[ \psi^2 = \text{Intensity} = 4A^2 \cos^2(\Delta k x - \Delta w t) \]

\[ |\psi|^2 = \text{intensity} = 4A^2 \cos^2(\Delta k x - \Delta w t) \]

Beats travel at speed \( v_{\text{group}} = v_{\text{G}} = \frac{\Delta w}{\Delta k} = \frac{dw}{dk} \)

For both light and matter waves:

\[ \frac{dw}{dk} = \left( \frac{\ln k}{m} \right) = \nu \quad \text{matter} \]

\[ \frac{1}{c} \quad \text{light} \]

To analyze waves: Fourier series + periodic functions

Consider a periodic function: \( f(x + L) = f(x) \)

=> I can write \( f(x) \) as a Fourier series, either

\[ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x} \]

\[ k_n = \frac{2\pi n}{L} \]

These exponential functions are orthogonal:

\[ \int_0^L e^{i(k_n - k_m) x} dx = 0 \quad \text{if } n \neq m \]
\[
\phi_n(x) = \frac{1}{\sqrt{L}} e^{i k_n x} \quad \text{are normalized and orthogonal.}
\]

\[
\Rightarrow \text{coefficients: } c_m = \frac{1}{L} \int_0^L f(x) e^{-i k_m x} \, dx
\]

Cases: \((\Rightarrow f(x) \text{ is real} \Rightarrow f^*(x) = f(x))\)
\[
\Rightarrow c_m^* = e^{-n} \quad c_0^* = c_0 = \text{real}
\]

\text{Parity: Cosine and sine have property:}
\[
\cos(k_n x) = + \cos(-k_n x) \quad \text{even parity}
\]
\[
\sin(k_n x) = - \sin(-k_n x) \quad \text{odd parity}
\]