Most sources of α-particles (He\(^{++}\)) are heavy nuclei that decay. A typical energy of the α-particle is kinetic 5 MeV. Rutherford scattering has α-particles scattering from a thin gold foil, specifically the gold nuclei. The α-particles and gold nuclei are much smaller than the size of atoms. An approximate formula for a nuclear radius is:

\[ r_N = (1.3 \times 10^{-15}) \left( A^{1/3} \right) \text{ (meter)} \]

where \( A \) = number of nucleons.

\[ \Rightarrow \text{ for gold } r_N \approx 7 \times 10^{-15} \text{ meter} \]

If the gold foil is thin enough, α-particles either go straight through or have one scattering event.

Because both the α-particle and nucleus are positively charged, the Coulomb force between them is repulsive and:

\[ F = \frac{Z \cdot Z' e^2}{4 \pi \varepsilon_0 r^2} \]

Rutherford scattering is a Kepler problem - \( \frac{1}{r^2} \) central force with center at one focus. The orbits are conic sections.

For example: ellipses

\[ 2ae \]

\[ 2a \]
2a e between foci
\( E = \text{eccentricity} \)
\( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \)

area of ellipse = \( 2 \int_{-a}^{+a} \sqrt{1 - \frac{x^2}{a^2}} \ b \ dx \)

In polar coordinates, with the origin at one focus:

\[ r + r' = 2a \]
\[ r'^2 = r^2 + 4a^2 \varepsilon^2 - 4a \varepsilon r \cos \phi = 4a^2 - 4ar + r^2 \]
\[ 4a^2 \varepsilon^2 - a \varepsilon r \cos \phi + ar - a^2 = 0 \]
\[ \Rightarrow r = \frac{a(1 - \varepsilon^2)}{1 - \varepsilon \cos \phi} \]
\[ \frac{1}{r} = A + B \cos \phi \]

All conic sections can be written in the form:
\[ \frac{1}{r} = A + B \cos \phi \]

Another example: Hyperbolas
\( \alpha \) is the angle of the asymptote w.r.t. the \( \hat{x} \)-axis. There are two branches:

\[
\begin{align*}
  r' - r &= 2a & \text{+ branch} \\
  r' - r &= -2a & \text{- branch}
\end{align*}
\]

Using the cosine law:

**+ branch:**

\[
 r'^2 = r^2 + 4az^2 - 4ar \cos \phi
\]

\[
 = 4a^2 + 4ar + r^2
\]

\[
 \Rightarrow a(\epsilon^2 - 1) = r(1 + \epsilon \cos \phi)
\]

\[
 \Rightarrow r = \frac{a(\epsilon^2 - 1)}{1 + \epsilon \cos \phi}
\]

**- branch:**

\[
 r = \frac{a(\epsilon^2 - 1)}{-1 + \epsilon \cos \phi}
\]

\[
 \Rightarrow \frac{1}{r} = A + B \cos \phi
\]

\[
 A = \frac{\pm 1}{a(\epsilon^2 - 1)}
\]

\[
 B = \frac{\epsilon}{a(\epsilon^2 - 1)}
\]

The impact parameter \( (b) \) is the perpendicular to the asymptote from the focus. The perpendiculars from \( F \) and \( F' \) are the same.

Also: \( \sin \alpha = \frac{b}{a \epsilon} \)

For the \((-\) branch \( r \to \infty \) as \( \cos \phi = \cos \alpha \approx \frac{1}{\epsilon} \).
So now we have a Kepler problem with a Coulomb force \[ k = \frac{Z Z' e^2}{r^2} \]

\[ F = \left( m \frac{d^2 x}{dt^2} \right) \hat{x} + \left( m \frac{d^2 y}{dt^2} \right) \hat{y} = \left( \frac{k}{r^2} \right) \hat{r} \]

If \( k > 0 \): repulsive  
If \( k < 0 \): attractive

Transform to polar coordinates:
\[ x = r \cos \theta \Rightarrow \dot{x} = \cos \theta \dot{r} - \sin \theta \dot{\theta} \]
\[ y = r \sin \theta \Rightarrow \dot{y} = \sin \theta \dot{r} + \cos \theta \dot{\theta} \]

\[
\Rightarrow \dot{x} = \ddot{r} \cos \theta - (r \dot{\theta}) \dot{\theta} - (r \cos \theta) \ddot{\theta}^2 - (r \sin \theta) \dot{\theta}^2 \\
\dot{y} = \ddot{r} \sin \theta + (r \dot{\theta}) \cos \theta - (r \sin \theta) \ddot{\theta}^2 + (r \cos \theta) \dot{\theta}^2 \\
\Rightarrow \ddot{r} \hat{r} + \ddot{\theta} \hat{\theta} = \left( \frac{k}{r^2} \right) \hat{r} \\
\]

Since there is no \( \dot{\theta} \) term to the force:
\[ 2 \dot{r} \dot{\theta} + r \ddot{\theta} = 0 \]
which is conservation of angular momentum
\[
\hat{L}_z = (mr^2 \dot{\theta}) \hat{z} \Rightarrow \frac{dL_z}{dt} = m \ddot{r} \dot{\theta} + r \ddot{\theta} = 0 \\
\]
So the force equation is:
\[ m \left( \ddot{r} - r \ddot{\theta}^2 \right) = \frac{k}{r^2} \]

where \( L = mr^2 \dot{\theta} = \text{constant} \Rightarrow \dot{\theta} = \frac{L}{mr^2} \)

\[ m \left( \ddot{r} - \frac{L^2}{m^2 r^3} \right) = \frac{k}{r^2} \]
Change of variables: Let \( u = \frac{1}{r} \)

\[ r = \left( \frac{-u^2}{m^2} \right) \frac{d}{dt} u \]

\[ u = \left( \frac{dy}{d\theta} \right) \frac{d}{d\theta} = \left( \frac{L}{m} \right) (u^2) \frac{d}{d\theta} \]

\[ \Rightarrow \frac{d}{d\theta} \frac{d}{d\theta} = \left( \frac{-L^2}{m^2} \right) (u^2) \frac{d}{d\theta} \left( \frac{1}{u^2} \right) (\frac{L}{m}) \frac{dy}{d\theta} = -L^2 \frac{d^2u}{d\theta^2} \]

\[ \Rightarrow m \left( \frac{L^2}{m^2} \frac{d^2u}{d\theta^2} - L^2 u^3 \right) = K u^2 \]

\[ \Rightarrow \left( \frac{L^2}{m^2} u^2 \right) \left( \frac{d^2u}{d\theta^2} + u \right) = -K u^2 \]

which is in the form: \( \ddot{u}(\theta) + u(\theta) = \frac{-K u^2}{L^2} \)

This looks like a driven harmonic oscillator.

The homogeneous equation is:

\[ \ddot{u}(\theta) + u(\theta) = 0 \]

\[ \Rightarrow u(\theta) = A_0 \cos \theta + B_0 \sin \theta \]

Particular solution to the inhomogeneous equation:

\[ u(\theta) = \frac{-K u^2}{L^2} \]

\[ u(\theta) = \frac{L^2}{u^2} = A_0 \cos \theta + B_0 \sin \theta = \frac{-K u^2}{L^2} \]

Recall general form for a conic section:

\[ \frac{1}{r} = A + B \cos \phi \]

\[ \Rightarrow B = 0 \Rightarrow A_0 = B \Rightarrow A = \frac{-K u^2}{L^2} \]
All we really need is:
\[ -\frac{mK}{L^2} = \frac{-1}{a(E^2-1)} \quad \text{and} \quad L = mv/b \]

\[ E^2 - 1 = \sec^2 \alpha - 1 = \tan \alpha \]

\[ a = \frac{b}{E \sin \alpha} = \frac{b}{\tan \alpha} \]

\[ \Rightarrow a(E^2 - 1) = b \tan \alpha \]

\[ \frac{mK}{L^2} = \frac{k}{mv^2b} \quad \Rightarrow \quad \text{Since } E_{\alpha} = \frac{1}{2} mv^2 \text{ is the particle kinetic energy} \]

\[ b = \frac{k \tan \alpha}{2E_{\alpha}} \]

\[ \Rightarrow \alpha = \frac{\pi}{2} - \frac{\Theta}{2} \]

\[ \tan \alpha = \cot \left( \frac{\Theta}{2} \right) \]

\[ b = \left( \frac{k}{2E_{\alpha}} \right) \cot \left( \frac{\Theta}{2} \right) \]

This is the link between the force constant \( K \), the scattering angle \( \Theta \), the \( \alpha \)-particle kinetic energy \( E_{\alpha} \) and the impact parameter \( b \).

\[ \Rightarrow \quad db = \left( \frac{k}{2E_{\alpha}} \right) \frac{1}{\sin^2 (\Theta/2)} \frac{d\Theta}{2} \]

\[ \Rightarrow \quad 2\pi b d\theta b = 2\pi \left( \frac{k^2}{4E_{\alpha}} \right) \frac{\cos (\Theta/2)}{\sin^3 (\Theta/2)} \frac{d\Theta}{2} \]

Solid angle \( d\Omega = 2\pi \sin \theta d\theta \)
\[
2\pi b d\phi = 2\pi \left( \frac{\kappa^2}{16E_x^2} \right) \frac{1}{\sin^4(\theta/2)} (\sin \theta \, d\theta)
\]

Pictorially:

\[ b = \left( \frac{Z_1 Z_2 e^2}{4\pi \varepsilon_0} \right) \left( \frac{1}{2E_x} \right) \cot \left( \frac{\theta}{2} \right) \]

Since the solid angle \( d\Omega = 2\pi \sin \theta \, d\theta = 4\pi \sin(\theta/2) \cos(\theta/2) \, d\theta \)

\[
\Rightarrow \cos \left( \frac{\theta}{2} \right) \, d\theta = \frac{d\Omega}{4\pi \sin(\theta/2)}
\]

\[
\Rightarrow d\sigma = \left( \frac{\kappa^2}{8E_x^2} \right) \frac{2\pi \, d\Omega}{4\pi \sin^4(\theta/2)} = \left( \frac{\kappa^2}{4E_x} \right) \frac{1}{\sin^4(\theta/2)} \, d\Omega
\]

This gives Rutherford's cross-section formula:

\[
\frac{d\sigma}{d\Omega} = \left( \frac{Z_1 Z_2 e^2}{4\pi \varepsilon_0} \right)^2 \left( \frac{1}{16E_x^2} \right) \frac{1}{\sin^4(\theta/2)} \quad \text{(dimensions: area)}
\]

For \( \alpha \)-particles on gold: \( Z_1 Z_2 = 2 \times 79 = 158 \)

other constants: \( h = 4 \times 10^{-34} \text{ J s} \), \( 2.27 \text{ eV nm} \)

For a 5 MeV \( \alpha \)-particle:

\[
\frac{d\sigma}{d\Omega} = \frac{1.29 \text{ barns}}{\sin^4(\theta/2)} \quad 1 \text{ barn} = 10^{-10} \text{ nm}^2
To integrate between two scattering angles:

\[
\Delta \sigma = 2\pi \int_{\theta_1}^{\theta_2} \left( 1.29 \text{ barns} \right) \frac{\sin \theta d\theta}{\sin^4(\theta/2)}
\]

\[
= 2 \left[ \frac{1}{\sin^2(\theta_2/2)} - \frac{1}{\sin^2(\theta_1/2)} \right]
\]

\[
\Rightarrow \Delta \sigma = (4\pi)(1.29) \text{ barns} \left[ \frac{1}{\sin^2(\theta_2/2)} - \frac{1}{\sin^2(\theta_1/2)} \right]
\]

You will see this written another way:

\[
\Delta \sigma = (4\pi)(1.29) \text{ barns} \left[ \cot^2(\theta/2) - \cot^2(\theta_1/2) \right]
\]

because:

\[
\left[ \frac{1}{\sin^2(\theta_2/2)} - \frac{1}{\sin^2(\theta_1/2)} \right] = \left[ \frac{\sin^2(\theta_2/2) + \cos^2(\theta_2/2)}{\sin^2(\theta_2/2)} - \frac{\sin^2(\theta_1/2) + \cos^2(\theta_1/2)}{\sin^2(\theta_1/2)} \right]
\]

\[
= \left[ \frac{\cos^2(\theta_2/2)}{\sin^2(\theta_2/2)} - \frac{\cos^2(\theta_1/2)}{\sin^2(\theta_1/2)} \right] = \left[ \cot^2(\theta/2) - \cot^2(\theta_1/2) \right]
\]

In measurements you measure the number of \( \alpha \)-particles scattered per gold nucleus per unit time as:

\[
\frac{dN}{dt} = \left[ \text{# incident per area per time} \right] \times \left[ \text{# gold/nuclei} \right] \times \Delta \sigma
\]

Measuring the incident flux is hard. Measuring the total incident current \( I_0 = \left( \# \text{ \( \alpha \)-particles} \right) \text{ per second} \) is easier.
\[
\frac{dN}{dt} = I_0 \left[ \frac{\text{gold nuclei}}{\text{area}} \right] d\Gamma = ?
\]

\[
\frac{\text{gold nuclei}}{\text{cm}^2} = \frac{\rho N_{A_v} t}{M_w}
\]

\( \rho = \text{density of gold (g/m}^3 \) 

\( N_{A_v} = 6.02 \times 10^{23} \)

\( M_w = \text{g/mole in 1 mole} \)

\( t = \text{thickness (cm.)} \)

\( M_w = 197 \text{ g/mole} \)

\( \rho = 19.3 \text{ g/m/cm}^3 \)

For \( t = 2 \mu m \)

\[
\frac{\rho N_{A_v} t}{M_w} = 1.18 \times 10^{19} \frac{\text{nuclei}}{\text{cm}^2} = 1.18 \times 10^{-5} \frac{\text{nuclei}}{\text{barn}}
\]

Note: (1) This experiment must be done in vacuum. Why? Because 5MeV \( \alpha \)-particles only go about 4 cm in air before losing all their kinetic energy to ionization.

(2) Range of 5MeV \( \alpha \)-particles in gold is about 3 \( \mu m \), so the foil should not be thicker than about 2 \( \mu m \).