Problem. Consider a particle of mass $m$ moving in a vertical $x - y$ plane along a curve $y = a \cos(2\pi x/\lambda)$. The free fall acceleration is $g$. First, consider its motion in terms of two coordinates $x$ and $y$:

a) Write down the Lagrangian of the system.

b) Obtain the Lagrange’s equation of motion with undetermined multipliers.

c) Define the forces of the constraint. What is the minimal velocity of a particle at the top of the curve when the force of constraint vanishes? Compare your expressions for the forces of constraint with the forces you would introduce to formulate the problem in terms of Newton’s second law.

Now, we choose a single generalized coordinate corresponding to horizontal coordinate $x$.

d) Write down the Lagrangian of the particle and Lagrange’s equation of motion.

e) Define the generalized momentum and write down the Hamiltonian of the particle.

f) Derive the canonical (Hamilton’s) equations of motion.

g) Demonstrate that the canonical equations of motion can be reduced to Lagrange’s equation of motion.

Solutions

a) The kinetic energy is $T = (m/2)(\dot{x}^2 + \dot{y}^2)$ and the potential energy is $U = mgy$. The Lagrangian is

$$L = \frac{m\dot{x}^2}{2} + \frac{m\dot{y}^2}{2} - mgy$$ (1)

b) The constraint can be defined as $\varphi(x, y) = 0$ with $\varphi(x, y) = y - a \cos(2\pi x/\lambda)$. The Lagrange’s equations are

$$-\frac{\partial L}{\partial x} + \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \alpha \frac{\partial \varphi}{\partial x} = 0;$$ (2)

$$-\frac{\partial L}{\partial y} + \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} - \alpha \frac{\partial \varphi}{\partial y} = 0$$ (3)

and reduce to

$$m\ddot{x} - \alpha a \frac{2\pi}{\lambda} \sin \frac{2\pi x}{\lambda} = 0;$$ (4)

$$mg + m\ddot{y} + \alpha = 0.$$ (5)

The constraint itself completes the system of three equations for three unknown functions, $x(t)$, $y(t)$ and $\alpha(t)$. Using the constraint, we can express $\alpha(t)$ in terms $x(t)$ and $y(t)$. For this purpose, we differentiate the constraint twice with respect to time:

$$\ddot{y} = -\left(\frac{2\pi}{\lambda}\right)^2 a \cos \frac{2\pi x}{\lambda} \dot{x}^2 - \frac{2\pi a}{\lambda} \sin \frac{2\pi x}{\lambda} \dot{x}.$$ (6)
We obtain
\[ \alpha = -mg + \left( \frac{2\pi}{\lambda} \right)^2 a \cos \frac{2\pi x}{\lambda} x^2 + \frac{2\pi a}{\lambda} \sin \frac{2\pi x}{\lambda} \ddot{x}. \]  
(7)

c) The forces of the constraint are
\[ F_x = -\alpha \frac{\partial \varphi}{\partial x} = -\frac{2\pi a}{\lambda} \alpha \sin \frac{2\pi x}{\lambda}, \]
\[ F_y = -\alpha \frac{\partial \varphi}{\partial y} = -\alpha, \]
(8)
(9)
where \( \alpha \) is defined by Eq. (7).

At the topmost point, \( \sin(2\pi x/\lambda) \). We have \( F_x = 0 \) and \( F_y = mg - (2\pi/\lambda)^2 a \cos(2\pi x/\lambda) \dot{x}^2 + 0 \). We have \( F_y = 0 \) for \( \dot{x} = \pm (\lambda/2\pi)/\sqrt{mg\alpha} \).

To compare with Newton’s equations, we have \( m \ddot{x} = N_x = N \sin \theta \) and \( m \ddot{y} = N_y - mg = N \cos \theta - mg \), where \( \tan \theta = dy/dx = (2\pi a/\lambda) \sin(2\pi x/\lambda) \). We have \( m \ddot{x} = N_y \tan \theta = N_y (2\pi a/\lambda) \sin(2\pi x/\lambda) \), cf. with Eq. (8).

d) We have \( \dot{y} = -(2\pi a/\lambda) \sin(2\pi x/\lambda) \dot{x} \) and the kinetic energy is
\[ T = \frac{m \dot{x}^2}{2} \left[ 1 + \left( \frac{2\pi a}{\lambda} \right)^2 \sin^2 \frac{2\pi x}{\lambda} \right], \]
(10)
while
\[ U = mga \cos \frac{2\pi x}{\lambda}. \]
(11)

We find
\[ L = T - U = \frac{m \dot{x}^2}{2} \left[ 1 + \left( \frac{2\pi a}{\lambda} \right)^2 \sin^2 \frac{2\pi x}{\lambda} \right] - mga \cos \frac{2\pi x}{\lambda}. \]
(12)
The Lagrange’s equation
\[ \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0 \]
(13)
is
\[ -m \ddot{x} \left[ 1 + \left( \frac{2\pi a}{\lambda} \right)^2 \sin^2 \frac{2\pi x}{\lambda} \right] - m \dot{x} \left( \frac{2\pi a}{\lambda} \right)^2 \frac{2\pi x}{\lambda} \frac{2\pi x}{\lambda} \frac{2\pi x}{\lambda} \frac{2\pi x}{\lambda} \sin \frac{2\pi x}{\lambda} \frac{2\pi x}{\lambda} \sin \frac{2\pi x}{\lambda} = 0, \]
(14)
or
\[ -m \ddot{x} \left( \frac{2\pi a}{\lambda} \right)^2 \frac{2\pi x}{\lambda} \sin \frac{2\pi x}{\lambda} - m \ddot{x} \left[ 1 + \left( \frac{2\pi a}{\lambda} \right)^2 \sin^2 \frac{2\pi x}{\lambda} \right] = 0. \]
(15)
e) The generalized momentum is
\[ p = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \frac{2\pi a}{\lambda} \left[ 1 + \left( \frac{2\pi a}{\lambda} \right)^2 \sin^2 \frac{2\pi x}{\lambda} \right]. \]
(16)
We express $H = p\dot{x} - L$ in terms of $x$ and $p$ variables by substitution

$$\dot{x} = \frac{p}{m} \left[ 1 + \left( \frac{2\pi a}{\lambda} \right)^2 \sin^2 \frac{2\pi x}{\lambda} \right]^{-1}.$$  \hspace{1cm} (17)

The Hamiltonian is

$$H = \frac{p^2}{2m} \left[ 1 + \left( \frac{2\pi a}{\lambda} \right)^2 \sin^2 \frac{2\pi x}{\lambda} \right]^{-1} + mga \cos \frac{2\pi x}{\lambda}. \hspace{1cm} (18)$$

or

f) $$\dot{x} = \frac{\partial H}{\partial p} \hspace{1cm} (19)$$

and gives the same result as Eq. (17), as expected.

$$\dot{p} = -\frac{\partial H}{\partial x} = \frac{p^2}{2m} \left[ 1 + \left( \frac{2\pi a}{\lambda} \right)^2 \sin^2 \frac{2\pi x}{\lambda} \right]^{-2} 2 \sin \frac{2\pi x}{\lambda} \cos \frac{2\pi x}{\lambda} + mga \sin \frac{2\pi x}{\lambda}. \hspace{1cm} (20)$$

g) From Eq. (16) we find

$$\dot{p} = m\ddot{x} \left[ 1 + \left( \frac{2\pi a}{\lambda} \right)^2 \sin^2 \frac{2\pi x}{\lambda} \right] + m\dot{x}^2 \left( \frac{2\pi a}{\lambda} \right)^2 2 \sin \frac{2\pi x}{\lambda} \cos \frac{2\pi x}{\lambda}, \hspace{1cm} (21)$$

and substituting it to Eq. (20) and using Eq. (16), we arrive to Eq. (15).