Problem 1. (15 points) Consider a damped harmonic oscillator of mass $m$ described by its equation of motion $\ddot{x} + 2\beta \dot{x} + \omega_0^2 x = \left(\frac{F_0}{m}\right) \sinh(\lambda t)$ for $t \geq 0$. Find $x(t)$ that satisfies initial conditions $x(0) = a$ and $\dot{x}(0) = 0$.

Problem 2. Consider a system described by the Lagrangian:

$$L(x_1, \dot{x}_1; x_2, \dot{x}_2) = \frac{m_1 \dot{x}_1^2}{2} + \frac{m_2 \dot{x}_2^2}{2} + \frac{M(\dot{x}_1 - \dot{x}_2)^2}{2} - \frac{k_1 x_1^2}{2} - \frac{k_2 x_2^2}{2}. \quad (1)$$

a) (5 points) Write down Lagrange’s equations of motion;
b) (10 points) Find the characteristic frequencies and normal modes of this system;
c) (5 points) Define the generalized momenta for the system and obtain the Hamiltonian;
d) (5 points) Write down the canonical equations of motion and describe their solution in the phase space.

Problem 3. Consider two identical, coupled oscillators. Let each of the oscillators be damped, and let each have the same damping parameter $\beta$. One oscillator is held at its equilibrium point, while the other is slowly displaced by distance $a$. The work of the force to perform this displacement is $W$. The two oscillators are simultaneously released.

a) (10 points) Determine normal modes and write down displacement of the oscillators as a function of time after they are released. Which mode has a longer relaxation time?
b) (10 points) Define the ratio of the total energy losses in each oscillator to the initial value of the excessive energy $W$ (total energy minus energy in equilibrium) after a long observation time. Represent the ratio as a function of the coupling strength $\kappa$ between the oscillators.

Problem 4. (15 points) Calculate the least required mass of fuel $m_f$ to place a satellite on a geostationary orbit from a geocentric (circular) orbit 200km above the Earth orbit, if the mass of the satellite, engine and fuel tanks is $m_l$. Consider a trajectory to the geostationary orbit as an ellipse with perigee at the initial geocentric orbit and apogee at the circular geostationary orbit. While the satellite is on a geocentric orbit, it fires a short pulse to switch to the elliptic part of its trajectory, and fires again at the apogee to move along a circular geostationary orbit. The exhaust velocity from the engine is $u = 3\text{km/s}$. Assume that the ideal rocket equation is applicable to describe the change in the satellite velocity during the engine pulses.

Problem 5. Calculate the cross-section for an attractive potential $U(r) = -\lambda/r^3$ to trap a particle moving initially with energy $E > 0$.

a) (10 points) Draw the effective potential and find its maximal value as a function of impact parameter $b$.
b) (5 points) Find the marginal value $b_m$ at which the particle is trapped. Sketch trajectories for $b > b_m$ and $b < b_m$ and argue that the cross-section to trap is $\sigma_{\text{trap}} = \pi b_m^2$.

Problem 6. (10 points) A rigid body consists of four identical masses $m$ placed at the vertices of a regular tetrahedron with edges of length $a$. Find the inertia tensor, principal moment of inertial and principal axes with respect to the center of mass of the system.