Laplace's equation and method of images (Ch. 2, Purcell)

Capacitance and Capacitors (Ch. 3, Purcell)

Method of Images

replace original problem by simpler problem that satisfies same boundary conditions in the region of interest

Example 1:

Point charge $q$ a distance $h$ away from infinite grounded conducting plane.

$\nabla \cdot \mathbf{E} = -\varepsilon_0 \frac{\partial \phi}{\partial z}$

$\sigma(x, y, 2) = \frac{q}{4\pi \varepsilon_0 \left[(x^2 + y^2 + (2+h)^2)^{3/2}\right]}

\nabla \cdot \mathbf{E} = \frac{q}{2\pi \varepsilon_0 \left[(x^2 + y^2 + h^2)^{3/2}\right]}

\text{Energy: Work done to bring } q \text{ from } \infty \text{ in field of } q'$
Example 2

Point charge and a grounded conducting sphere

boundary conditions: $\phi(r = R) = 0$

$\phi \to 0$ for $r^2 \gg a^2$

Replace by image problem $\Rightarrow q'$ a distance $b$ away from center of sphere

$a > R > b$
$q'$ inside sphere

$q' + b$ are to be determined from $\phi(r = R) = 0$ condition.

\[
\phi(r, \theta) = \frac{1}{4\pi\varepsilon_0} \left[ \frac{q}{(r^2 + a^2 - 2ar\cos\theta)^{1/2}} + \frac{q'}{(r^2 + b^2 - 2br\cos\theta)^{1/2}} \right]
\]

$\phi(r = R) = 0$
$\theta = 0 \quad + \quad \theta = \pi$

\[
\frac{q}{a-R} + \frac{q'}{R-b} = 0
\]

\[
\frac{q}{R+a} + \frac{q'}{R+b} = 0
\]

solve to obtain

\[
q' = -\frac{q}{a} \quad b = \frac{R^2}{a}
\]

Note: image charge magnitude not equal to original charge in this example.
Capacitance and Capacitors

Consider an isolated conductor with charge $Q$.

- Potential $\phi_0$ depends on shape/size of conductor.
- Is proportional to $Q$.

$$Q = C \phi_0$$

Capacitance is the measure of the capacity to store charge for a given potential difference.

- Dependent on shape/size of conductor.

E.g. Sphere of radius $a$:

$$\phi_0 = \frac{Q}{4\pi\varepsilon_0 a} \quad \text{at } r=a \quad \text{(SI units)}$$

$$C = 4\pi\varepsilon_0 a$$

$$\Rightarrow [C] = \frac{\text{Coulomb}}{\text{volt}} = \text{Farad}$$

($\varepsilon_0$ is a large unit.

(Usually have µF or pF.)
cgs units: $C_{\text{sphere}} = a \quad [C] = \text{cm}$

**Capacitor** is a device consisting of two conductors that are oppositely charged $(Q + -Q)$ at potential difference $\Delta \phi$.

The capacitance of the capacitor is

$$C = \frac{Q}{\Delta \phi}$$

$q = \text{magnitude of charge on either conductor}$

Often just write $C = Q/\phi$ or $C = \epsilon/\epsilon_0 \quad \epsilon_0 = \text{volt} / \text{cm}$

To compute $C$, put charges $Q + -Q$ on conductors.

- Either (i) determine $E$, then $\Delta \phi = -\int E \cdot ds$
  - Focus on this course on (i).
- Or (ii) determine $\phi$ directly.

Prototype: **Parallel plate capacitor**

\[ \begin{array}{c}
\text{[Diagram of parallel plate capacitor with charges, area $A$, separation $d$]} \\
\text{Assume $\sqrt{A} \gg d$ \quad can \ reject \ edge \ effects.} \\
E = \frac{V}{\epsilon_0} \quad \rightarrow \quad |\Delta \phi| = \frac{V}{\epsilon_0} \quad C = \frac{Q}{V} = \frac{\epsilon A}{\epsilon_0 d} = \epsilon_0 A/d
\end{array} \]
We see that capacitance is a geometrical quantity.

**Example:** Coaxial cylinders

- **Q** on inner cylinder of radius **a**
- **Q** on outer cylinder of radius **b**

Assume a long cylinder of length \( L \gg a, b \).

\[
\vec{E}_{a < r < b} = \frac{\lambda}{2\pi \varepsilon_0 r}
\]

\[
\Rightarrow \Delta \Phi = -\int_{a}^{b} \vec{E} \cdot d\vec{r} = -\frac{\lambda}{2\pi \varepsilon_0} \ln \frac{b}{a}
\]

\[
C = \frac{Q}{\Delta \Phi} = \frac{\lambda L}{2\pi \varepsilon_0 \ln \frac{b}{a}} = \frac{2\pi \varepsilon_0 \lambda L}{\ln \frac{b}{a}}
\]

Comment: one can consider the potentials + charges on several conductors (> 2)

Example (Purcell section 3.6) 3 conductors enclosed by conducting shell

\[
Q_1 = C_{11} \phi_1 + C_{12} \phi_2 + C_{13} \phi_3
\]

\[
Q_2 = C_{21} \phi_1 + C_{22} \phi_2 + C_{23} \phi_3
\]

\[
Q_3 = C_{31} \phi_1 + C_{32} \phi_2 + C_{33} \phi_3
\]

\[
C_{ij} : \text{coefficients of capacitance}
\]

\[
C_{ij} = C_{ji}
\]
Energy stored in capacitor

Capacitor stores charge \( \rightarrow \) stores energy
(work must be done to charge a capacitor)

Consider process of charging capacitor: transfer \( dQ \) from \( \Theta \) to \( \Theta' \) plate

Work done:

\[
\delta W = \left| \frac{\Delta \phi}{dQ} \right| dQ = \frac{Q dQ}{C}
\]

\[
\Rightarrow W = \frac{1}{C} \int_{Q_i=0}^{Q_f} Q dQ = \frac{Q_f^2}{2C}
\]

\[
\Rightarrow \text{potential energy stored in capacitor is}
\]

\[
U = \frac{1}{2} \frac{Q_f^2}{2C} = \frac{1}{2} C (\Delta \phi)^2 = \frac{1}{2} C \Delta \phi \Delta \phi
\]

How to charge it: \( \rightarrow \) eg connect it to a battery

Work done by battery: \( Q V = Q \Delta \phi \)

Battery = source of potential difference (voltage) \( \Delta \phi \)