Physics 248  Lecture 7

Topics

introduce the curl of a vector function \{ Purcell, Chapter 2
2 Maxwell's eqns for electrostatic fields

begin conductors in electrostatic fields \{ Purcell, Chapter 3

Curl

Previously, we discussed the divergence of a vector field \( \vec{F} \)

\[
\text{div} \, \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad \text{(Cartesian coordinates)}
\]

define \( \text{div} \, \vec{F} \) as the limit of (the flux of \( \vec{F} \) through a closed surface \( S \)) divided by (the volume \( V \) enclosed by \( S \)) as the volume \( V \) shrinks:

\[
\text{div} \, \vec{F} = \lim_{V \to 0} \frac{1}{V} \oint_S \vec{F} \cdot d\vec{a}
\]

+ showed \( \text{div} \, \vec{F} = \nabla \cdot \vec{F} \) with an example in Cartesian coordinates.
This led us to the divergence theorem:

\[ \oint_S \mathbf{F} \cdot d\mathbf{a} = \int_V \text{div} \mathbf{F} \, dV = \int_V \nabla \cdot \mathbf{F} \, dV \]

where \( S \) is the surface that bounds volume \( V \).

Using this, we can apply it to Gauss's law in integral form:

\[ \oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{q_{\text{enclosed}}}{\varepsilon_0} = \int_V \rho \, dV \frac{1}{\varepsilon_0} \]

\[ \int_V \nabla \cdot \mathbf{E} \, dV = \int_V \rho \, dV \frac{1}{\varepsilon_0} \Rightarrow \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]

This is the differential form of Gauss's law.

Using the fact that \( \mathbf{E} = -\nabla \phi \), we then obtain Poisson's equation:

\[ \nabla^2 \mathbf{E} = \nabla \cdot (-\nabla \phi) = \frac{\rho}{\varepsilon_0} \Rightarrow \nabla^2 \phi = -\frac{\rho}{\varepsilon_0} \]

where \( \nabla^2 = \nabla \cdot \nabla \) is the Laplacian operator.

For \( \rho = 0 \), we have Laplace's equation:

\[ \nabla^2 \phi = 0 \]
Now let us turn to the **curl** \( \text{curl } \vec{F} = \nabla \times \vec{F} \). This is a vector quantity. Together \( \text{div } \vec{F} + \text{curl } \vec{F} \) specify \( \vec{F} \) uniquely. (Helmholtz's theorem)

From \( \text{curl } \vec{F} = \nabla \times \vec{F} \),

we can write it out in Cartesian coordinates:

\[
\nabla \times \vec{F} = \hat{x} \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{y} \left( \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) + \hat{z} \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)
\]

The curl is a "sideways" derivative: depends on the variation of each component of \( \vec{F} \) with respect to the other variables.

(Compare the divergence, which measures the rate of change of each component with respect to its associated variable.)
Now define curl more systematically:

relate it to circulation (similar to how divergence is related to flux)

i.e. line integral of \( \vec{F} \) about a closed path \( C \)

\[
\Gamma = \oint_C \vec{F} \cdot d\vec{s}
\]

circulation

As before, subdivide \( C \) into smaller closed paths:

no contribution from center piece, so

\[
\Gamma = \Gamma_1 + \Gamma_2 = \oint_{C_1} \vec{F} \cdot d\vec{s}_1 + \oint_{C_2} \vec{F} \cdot d\vec{s}_2
\]

hence do it \( N \) times →

\[
\oint_{C} \vec{F} \cdot d\vec{s} = \sum_{i=1}^{N} \oint_{C_i} \vec{F} \cdot d\vec{s}_i = \sum_{i=1}^{N} \Gamma_i
\]

To obtain a quantity characteristic of \( \vec{F} \) in a local neighborhood,

consider ratio of \( \Gamma / \text{loop area} \)
Complication: area $\mathbf{a}$ is a vector

$\mathbf{a} = a_i \hat{n}$

direction: right-hand rule

Note that we get a vector quantity because limit taken will depend on orientation $\hat{n}$ of the patch.

$\text{curl} \mathbf{F}$ (like $\text{div} \mathbf{F}$) is a local function

direction of $\text{curl} \mathbf{F}$ at each point is normal to plane in which the circulation is maximum

magnitude of $\text{curl} \mathbf{F}$ is limiting value of circulation/area in this plane around pt.

$Γ = \oint C \mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^{N} (\mathbf{a}_i \cdot (\mathbf{F} \cdot \hat{n}))$

as $N\to \infty$ and $a_i \to 0$,

$Γ = \oint C \mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^{N} a_i \mathbf{curl} \mathbf{F} \cdot \mathbf{R}_i = \int_{S} \mathbf{curl} \mathbf{F} \cdot d\mathbf{a}$
This is Stokes' theorem:

\[ \oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \text{curl} \mathbf{F} \cdot d\mathbf{a} \]

where \( S \) is the surface bounded by the curve \( C \).

Now we need to show that \( \text{curl} \mathbf{F} \) defined in this way is equal to \( \nabla \times \mathbf{F} \).

As in the case of divergence, easiest to show this using Cartesian coordinates.

Consider vector field \( \mathbf{F} = \mathbf{F}(x,y,z) \). Take the line integral about a path in the \( xy \) plane (\( \mathbf{\hat{n}} = \mathbf{\hat{z}} \)).

We want \( \oint \mathbf{F} \cdot d\mathbf{s} \) about this path.

Upper + lower part in \( xy \) plane (legs 1 + 3):

\[ -\Delta x \frac{\partial F_x}{\partial y} \left( x + \Delta x, y + \Delta y \right) + \Delta x \frac{\partial F_x}{\partial y} \left( x + \Delta x/2, y \right) \approx -\Delta x \Delta y \frac{\partial^2 F_x}{\partial y^2} \]
\[ \Delta y \frac{F_y(x+\Delta x, y+\Delta y/2) - F_y(x, y+\Delta y/2)}{\Delta y} \approx \Delta x \frac{\partial F_y}{\partial x} \]

\[ \Rightarrow \Gamma = \oint C \vec{F} \cdot d\vec{s} = \Delta x \Delta y \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \]

\[ \text{area} \quad (\text{curl} \vec{F})_z \]

Can do similar exercise for paths in yz and xz plane to get \((\text{curl} \vec{F})_{x,y,z} : \)

\[ \text{curl} \vec{F} = \hat{x} \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{y} \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{z} \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \]

which is just \( \nabla \times \vec{F} \).

Consider an electrostatic field \( \vec{E} \): What is \( \nabla \times \vec{E} \)?

Clearly \( \nabla \times \vec{E} = 0 \) since \( \int \vec{E} \cdot d\vec{s} \) is independent of path!

This is one of Maxwell's equations \( \rightarrow \) but will get modified for general case i.e. \( \nabla \times \vec{E} = 0 \) only for static case.
\[ \vec{E} = -\nabla \phi \quad \nabla \times (\nabla \phi) = 0 \quad \text{curl of a gradient is zero} \]

note also: \( \nabla \cdot (\nabla \times \vec{F}) = 0 \) \( \text{divergence of a curl is zero} \! \)

Summarize Maxwell's equations so far:

\[ \nabla \cdot \vec{E} = \rho / \varepsilon_0 \quad \nabla \times \vec{E} = 0 \]

Conductors In Electrostatic \( \vec{E} \) fields (Chapter 3, Purcell)

Two categories of materials: \underline{conductors} and \underline{insulators}

\[ \uparrow \quad \uparrow \quad \text{essentially: have mobile charges,} \]
\[ \text{(carry currents, ...)} \quad \text{changes "fixed."} \]

Consider conductors in a \underline{static} situation \( \rightarrow \)

must have \( \vec{E} = 0 \) inside conductor, i.e. \( \vec{E}_{\text{in}} = \vec{E}_{\text{ind}} + \vec{E}_{\text{ext}} = 0 \)