Topics:

- continuous dielectrics (Purcell, Ch 10)
  dynamic fields → displacement current
  waves in dielectrics

- magnetic materials (Purcell, Ch 11)
  magnetic dipoles
  magnetic susceptibility

Last time → discussed properties of insulating materials (dielectrics) with dielectric constant \( K \) assumed: linear, isotropic

Dipole moment:

\[ \vec{P} = \alpha \vec{E} \]

\[ \vec{P} = \text{dipole moment}/\text{volume} = \varepsilon_0 \chi_e \vec{E} \]

Polarization of material → bound charges:

\[ \sigma_b = \vec{P} \cdot \hat{n} \]

\[ \rho_b = -\nabla \cdot \vec{P} \]

\( \vec{E} \) field sourced by both free + bound charges.
introduced $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$

$\Rightarrow \mathbf{D} = \varepsilon_0 \mathbf{E} = \varepsilon \mathbf{E}$
since $\mathbf{P} \propto \mathbf{E}$

\[ \nabla \cdot \mathbf{D} = \rho_{\text{free}} \quad \text{useful instead of} \quad \nabla \cdot \mathbf{E} = \rho / \varepsilon_0 \]
\[ \nabla \times \mathbf{E} = \mathbf{0} \]

$\Rightarrow$ field of free point charge immersed in dielectric:

\[ \mathbf{D} = \frac{q}{4\pi \varepsilon \ R^2} \Rightarrow \mathbf{E} = \frac{\mathbf{D}}{\varepsilon} = \frac{q}{4\pi \varepsilon_0 \ R^2} \hat{r} = \frac{\mathbf{E}_{\text{vac}}}{K} \]

weaker field in dielectric

Macroscopic view. Microscopics very complicated! great essay in sec. 10-12
bottom line: polarization affected by thermal effects, quantum effects $\Leftrightarrow$ physics 249!

So far, just static fields →
now consider dynamical fields
What if external $E$ field is changing with time?

- depends on response time
- nonpolar \(\rightarrow\) similar to static case
- polar \(\rightarrow\) can vary \(\rightarrow\) get $\kappa \varepsilon (\omega)$

When polarization $\overrightarrow{p}$ changes with time \(\rightarrow\) electric current

\[
\overrightarrow{J} = \frac{d \overrightarrow{p}}{dt}
\]

Sources magnetic fields!

\[
\nabla \times \overrightarrow{B} = \mu_0 \varepsilon_0 \frac{d \overrightarrow{E}}{dt} + \mu_0 \frac{d \overrightarrow{p}}{dt} + \mu_0 \overrightarrow{J}_{\text{free}} = \mu_0 \left( \varepsilon \frac{d \overrightarrow{E}}{dt} + \overrightarrow{J}_{\text{free}} \right)
\]

\[
= \mu_0 \left( \frac{d \overrightarrow{B}}{dt} + \overrightarrow{J}_{\text{free}} \right)
\]

\(\uparrow\) "displacement current"

Later we'll further modify this to include effects of magnetized matter.

Now, consider an EM wave in a dielectric (unbounded)

Assume perfect insulator: $\overrightarrow{J}_{\text{free}} = 0$

\[\rho_{\text{free}} = 0\]

Also take $\rho_0 = 0$ for simplicity.
\[ \nabla \cdot \vec{E} = 0 \quad \nabla \cdot \vec{B} = 0 \]
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \]

Wave eqn:
\[ \nabla \times \nabla \times \vec{E} = -\nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = \nabla \times (-\frac{\partial \vec{B}}{\partial t}) = -\frac{\partial}{\partial t} (\nabla \times \vec{B}) \]
\[ \Rightarrow -\nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \]
\[ \Rightarrow \nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{Similarly,} \quad \nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \]

\[ \Rightarrow \text{wave eqn, speed of wave is} \]
\[ v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0 k}} = \frac{c}{\sqrt{k}} = \frac{c}{n} \]
index of refraction

So for a wave solution
\[ \vec{E} = \vec{E}_0 \text{ Re } e^{i(k \cdot \hat{r} - \omega t + \phi)} \]
\[ \vec{B} = \vec{B}_0 \text{ Re } e^{i(k \cdot \hat{r} - \omega t + \phi)} \]

\[ \omega = \nu k \quad E_0 = \frac{c B_0}{\sqrt{k}} \quad \hat{k}_1 \vec{E}_0, \vec{B}_0 \text{ mutually } \perp \]
Magnetic Fields in Matter (Purcell, Ch 11)

Now consider magnetized matter.

Several options:
- Permanent magnets (e.g., ferromagnets)
- Induced magnetization with external \( \vec{B} \)
  - 2 options:  
    - Magnetization \( \parallel \vec{B} \) (paramagnets)
    - Magnetization \( \perp \vec{B} \) (diamagnets)

Here since \( \nabla \cdot \vec{B} = 0 \) always,

the first important "multipole" is once again a dipole (a magnetic dipole).

Prototype: Current loop.

Magnetic dipole moment
\[
\vec{m} = I \vec{A} \quad \text{area}
\]

Then vector potential \( \vec{A}(r) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2} \)

with \( \vec{B} = \nabla \times \vec{A} \), we see again the dipole field for the far zone

\( r \gg \delta \)

(note: in cgs units, extra \( \frac{1}{c} \) in \( \vec{m} \))

etc.
Consider the magnetic dipole in an external $\mathbf{B}$ field:

if $\mathbf{B}$ uniform, net $\mathbf{F} = 0$

but nonzero net torque:

$$\mathbf{\tau} = \mathbf{m} \times \mathbf{B}$$

potential energy

$$U = -\mathbf{m} \cdot \mathbf{B}$$

in direct analogy to electric case.

nonuniform $\mathbf{B}$:

$$\mathbf{F} = (\mathbf{m} \cdot \nabla) \mathbf{B}$$

(again analogous to electric case)

Now the question is how the material responds to an external $\mathbf{B}$ field, given this knowledge. as stated: in some cases get induced dipole moment parallel to field, in other cases it is antiparallel.

Consider electric currents in atoms.

take "Bohr model"

e$^-$ orbiting nucleus,

"current loop"

$$\mathbf{I} = \frac{e\nu}{2\pi r} \quad \Rightarrow \quad m = \frac{\pi r^2 I}{2} = \frac{e\nu r}{2}$$

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in terms of the angular momentum \( L = \text{MeVr} \),

\[
\vec{m} = -\frac{1}{2me} \vec{L}
\]

\( \rightarrow \) general relation (\( \vec{L} \) is constant of motion)

in a material \( \rightarrow \) many such current loops, in all directions

external \( \vec{B} \) then selects out a given direction

Magnetization \( \vec{M} = \frac{\text{magnetic dipoles}}{\text{volume}} \)

For this orbital motion, result is that \( \vec{M} \) is antiparallel to \( \vec{B} \)

due to quantum mechanical effects (diamagnetism)

\( \vec{e} \) also possesses angular momentum that is unrelated to orbital motion \( \rightarrow \)

"spin" \( \vec{S} \) magnitude \( \frac{\hbar}{2} \) \( \leftarrow \) "reduced Planck const" \( \hbar = \frac{1}{2\pi} \ (6.63 \times 10^{-34} \ \text{J.s}) \)

Purely QM effect!
\[ \vec{m}_s = -\frac{1}{m_e} \frac{1}{N} \vec{S} \]
\[ \vec{T} = \vec{m}_s \times \vec{B} \] acts to line up (unpaired) spins
\[ \Rightarrow \text{leads to } \vec{M} \parallel \vec{B} \] (paramagnetism)

terms of magnets: alignment persists even w/o external field (more shortly.)

The magnetization \( \vec{M} \) is the source of bound current densities,
much like the polarization \( \vec{P} \) is the source of bound charge densities.

They are:
\[ \vec{J}_b = \vec{\nabla} \times \vec{M} \] (volume current density)
\[ \vec{K}_b = \vec{M} \times \hat{n} \] (surface current density)

These current densities source \( \vec{B} \) fields, as do their "free" counterparts
\[ \vec{J}_{\text{free}}, \vec{K}_{\text{free}} \].

Again, it's useful to define a quantity \( \vec{H} \) whose curl is sourced by free currents only.
This quantity is:

\[ \vec{H} = \frac{\vec{B}}{\mu_0} \]

Units: \( [\vec{H}] = \text{A/m} \)

(e.g., oersted \( \rightarrow 1 \text{oersted} = \frac{10^3}{4\pi} \text{A/m} \))

\[ \nabla \times \vec{B} = \mu_0 \left( \vec{J}_f + \vec{J}_b \right) + \mu_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \left( \vec{J}_f + \nabla \times \vec{M} \right) + \mu_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \]

\[ \Rightarrow \nabla \times \vec{H} = \vec{J}_f + \epsilon \frac{\partial \vec{E}}{\partial t} \]

in absence of displacement current, \( \nabla \times \vec{H} = \vec{J}_f \)

\[ \Rightarrow \int \vec{H} \cdot d\vec{l} = \text{I}_\text{free enclosed} \] modified Ampere's law

For linear and isotropic materials, traditional to write

\[ \vec{M} = \chi_m \vec{H} \]

magnetic susceptibility

\[ \Rightarrow \vec{B} = \mu_0 \left( \vec{H} + \vec{M} \right) \]

\[ = \mu_0 \left( 1 + \chi_m \right) \vec{H} = \mu_0 \chi_m \vec{H} = \mu \vec{H} \]