Lec 35: Multiple source interference + circular aperture diffraction + single-slit diff.

last time: 2-source interference + phase shifter

today: multiple source interference + circular aperture diffraction + single-slit diff.
An array of sources

A_1 = A_2 = \ldots = A_N = A \quad \text{for the "amplitudes"}
\phi_1 = \phi_2 = \ldots = \phi_N = 0 \quad \text{for the phases}

r_1 \approx r_2 \approx \ldots \approx r_N \approx \bar{R}

\hat{E} \approx \frac{2}{\bar{R}} \left[ \frac{A}{\bar{R}} \cos(\omega t - kr_1) + \frac{A}{\bar{R}} \cos(\omega t - kr_2) + \ldots + \frac{A}{\bar{R}} \cos(\omega t - kr_N) \right]

= \frac{2}{\bar{R}} \frac{A}{\bar{R}} \text{Re} \left[ e^{i(\omega t - kr_1)} + e^{i(\omega t - kr_2)} + \ldots + e^{i(\omega t - kr_N)} \right]

= \frac{2}{\bar{R}} \frac{A}{\bar{R}} \text{Re} \left[ e^{i(\omega t - kr_1)} \left( 1 + e^{-ik(r_2 - r_1)} + e^{-ik(r_3 - r_1)} + \ldots + e^{-ik(r_N - r_1)} \right) \right]

r_2 - r_1 \approx d \sin \psi \quad r_3 - r_1 = r_3 - r_2 + r_2 - r_1 \approx 2d \sin \psi \quad r_N - r_1 = (N-1)d \sin \psi

Let \quad \delta \equiv k d \sin \psi

\hat{E} = \frac{2}{\bar{R}} \frac{A}{\bar{R}} \text{Re} \left[ e^{i(\omega t - kr_1)} \left( 1 + e^{-i\delta} + e^{-2i\delta} + \ldots + e^{-(N-1)i\delta} \right) \right]

= \frac{2}{\bar{R}} \frac{A}{\bar{R}} \text{Re} \left[ e^{i(\omega t - kr_1)} \frac{1 - e^{-iN\delta}}{1 - e^{-i\delta}} \right]
As you will show in your HW, this yields for the intensity,

\[ I = \frac{E_0 c A^2}{2 R^2} \left[ \frac{\sin \left( \frac{N \theta}{2} \right)}{\sin \left( \frac{\theta}{2} \right)} \right]^2 \]

Since a single source radiation yields \( I_0 = \frac{E_0 c A^2}{2 R^2} \),

\[ I = I_0 \left[ \frac{\sin \left( \frac{N \theta}{2} \right)}{\sin \left( \frac{\theta}{2} \right)} \right]^2 \]

As you will show in your HW,

\[ \lim_{N \to \infty} \left[ \frac{\sin \left( \frac{N \theta}{2} \right)}{\sin \left( \frac{\theta}{2} \right)} \right]^2 = N^2 \quad \text{for } N \in \mathbb{Z} \]

e.g.

\[ N = 10 \]

Q: How many \( \delta \) values correspond to values of zero intensity?
With large $N$, one can pinpoint a region of sky with great accuracy.

Full width: $\frac{N\delta^+}{2} = \pi, \quad \frac{N\delta^-}{2} = -\pi$

\[
\delta^+ - \delta^- = \frac{4\pi}{N} = k d (\sin \psi_+ - \sin \psi_-)
\]

\[
\approx k d (\psi_+ - \psi_-) = k d \Delta \psi
\]

\[
\Delta \psi \approx \frac{4\pi}{N} \frac{1}{kd} = \frac{2\lambda}{Nd}
\]

**Diffraction**

Diffraction $\approx$ interference due to the screen "sources" cancelling the incident radiation.

Rigorously, one needs to solve Maxwell's equations subject to absorptive boundary conditions on the screen.

Just as our approximation of 2-slits being composed of two radiating current elements in phase, one can approximate diffraction as due to superposition of "effective" radiating point sources "composing" the aperture only.
electric field strength

\[ E \approx \text{Re} \left\{ \sum_n A_n \frac{e^{i(kr - kr)}}{r_n} \right\} \]

- How polarization enters the sum is left ambiguous
- Radiation far away from the aperture is implicit
- How \( A_n \) relates to the incident plane wave amplitude requires an analysis starting from Maxwell equations that is beyond the scope of this course

\[ E = \text{Re} \left\{ \frac{i k}{2 \pi} \int_{\text{aperture}} \tilde{A} \frac{e^{i(kr - kr)}}{r} \right\} \]

where \( E_{\text{incident}} = \text{Re} \{ \tilde{A} e^{i(kr - kr)} \} \)

- Sufficient for relative amplitude analysis far away (i.e. far \( \equiv r \gg \Theta(10\lambda) \))
- For plane-waves incident at an angle \( \perp \) to the aperture \( \tilde{A} \) is \((x,y)\) independent

Note dimensions:

- \( \tilde{A} (\neq \tilde{A}_n) \) here has units of electric field
- \( [k] = [\text{length}]^{-1} \) & \( [\frac{1}{r}] = [\text{length}]^{-1} \) cancel \( [dxdy] = [\text{length}]^{-2} \)
example  Diffraction by a circular aperture

Let a plane wave \( E = \text{Re} \{ \tilde{A} e^{i(\omega t - kr)} \} \) be incident normally on a circular aperture. What is the intensity at point \( P \) on the \( z \)-axis far from the aperture?

\[
E = \text{Re} \left\{ \frac{i k}{2\pi} \int_{\text{aperture}} dx \, dy \, \tilde{A} \, \frac{e^{i(\omega t - kr)}}{r} \right\}
\]

where \( r = \sqrt{x^2 + y^2 + z^2} \)

\[
= \text{Re} \left\{ \frac{i k}{2\pi} \tilde{A} \int_0^{2\pi} d\theta \int_0^a dp \, p \, e^{i(\omega t - kr)} \frac{1}{\sqrt{p^2 + z^2}} \right\}
\]

With \( z \)-fixed,

\[
p^2 + z^2 = r^2 \Rightarrow \quad 2p \, dp = 2\rho \, d\rho
\]

\[
E = \text{Re} \left\{ \frac{i k}{2\pi} \tilde{A} e^{i\omega t} \int_0^{a^2+z^2} dr \, e^{-i kr} \right\}
\]
As you will show in HW,

\[
I = 2 \varepsilon_0 c A^2 \sin^2 \left( \frac{k a^2}{4 \varepsilon} \right)
\]

\[
\sqrt{a^2 + z^2} = z \sqrt{1 + \left( \frac{a^2}{z^2} \right)^2} = z \left( 1 + \frac{1}{2} \left( \frac{a^2}{z^2} \right)^2 + \cdots \right)
\]

\[
= z + \frac{1}{2} \frac{a^2}{z^2}
\]

Maximum with largest \( z \) is \( a/\lambda \)

\[
\frac{k a^2}{4 \varepsilon} = \frac{\lambda}{2} \Rightarrow \frac{4 \varepsilon}{k} \frac{a^2}{\lambda^2} = \frac{1}{z} \Rightarrow z = \frac{D^2}{\lambda^2} \quad D = 2a.
\]

e.g. \( a = 1 \) \( \text{mm} \) \( \lambda = 5000 \) \( \text{Å} \)

\[
\frac{I}{2 \varepsilon_0 c A^2}
\]

Fresnel region is complicated

Fraunhofer \( z \approx \frac{2D^2}{\lambda} \)

Fresnel region is simple
Single-slit diffraction using \( N \)-source interference in the limit \( N \to \infty \)

\[
I = I_0 \frac{\sin^2 \left( \frac{Nkd}{2} \sin \Psi \right)}{\sin^2 \left( \frac{kd}{2} \sin \Psi \right)}
\]

Since \( D = N d \),

\[
I = I_0 \frac{\sin^2 \left( \frac{Dd}{2} \sin \Psi \right)}{\sin^2 \left( \frac{kd}{2} N \sin \Psi \right)}
\]

Take \( N \to \infty \) keeping \( D \) fixed; since \( \sin \left( \frac{\beta}{N} \right) \to \frac{\beta}{N} \) as \( N \to \infty \)

\[
I \approx I_0 N^2 \frac{\sin^2 \beta}{\beta^2} \text{ where } \beta = \frac{Dk}{2} \sin \Psi
\]

Let \( I_{\Psi = 0} \equiv I_0 N^2 \) corresponding to \( \Psi = 0 \) intensity

\[
I = I_{\Psi = 0} \frac{\sin^2 \beta(\Psi)}{\beta^2(\Psi)}
\]