Lecture 32: Relativistic effect examples

Last time: \( F^\mu \rightarrow \sum_{\alpha, \beta=0}^3 \Lambda^\mu_\alpha (\tilde{\nu}) \Lambda^\nu_\beta (\tilde{\nu}) F_{\alpha\beta} \)

today:
- How plane waves transform under Lorentz transform
- Electric field of a moving point charge
Example

Suppose an EM plane wave travels in the $\hat{x}$-direction:

$$\textbf{E} = E_0 \hat{y} \cos(k x - \omega k t)$$
$$\textbf{B} = \frac{E_0}{c} \hat{z} \cos(k x - \omega k t)$$

What does the wave look like in the frame moving with velocity $v \hat{x}$ with respect to the original frame?

Using

$$\Gamma^{\mu \nu} \rightarrow \sum_{\alpha \beta = 0}^{3} \Lambda^\alpha_\mu \Lambda^\beta_\nu F^{\alpha \beta},$$

you can show

\begin{align*}
E'_x &= E_x \\
E'_y &= \gamma (E_y - v B_z) \\
E'_z &= \gamma (E_z + v B_y) \\
B'_x &= B_x \\
B'_y &= \gamma (B_y + v E_z) \\
B'_z &= \gamma (B_z - v E_y)
\end{align*}

\[\therefore E'_x = 0, \quad B'_x = 0 \quad \text{since no longitudinal field} \]
\[E'_z = 0, \quad B'_y = 0 \quad \text{since only one-polarization}. \]

\begin{align*}
E'_y &= \gamma (E_0 - v E_y) \cos(k x - \omega k t) \\
B'_z &= \gamma (E_0 - v E_y) \cos(k x - \omega k t)
\end{align*}

Note $E'_y = B'_z = 0$ when $v = 1 \Rightarrow$ vanishes!
What does the $\cos (kx - kt)$ turn into when expressed in terms of $(t', x')$?

\[ t' = \gamma t - \gamma v x \quad \Rightarrow \quad t = \gamma t' + \gamma v x' \]
\[ x' = \gamma x - \gamma v t \quad \Rightarrow \quad x = \gamma x' + \gamma v t' \]

\[ kx - kt = k(\gamma x' + \gamma v t') - k(\gamma t' + \gamma v x') \]
\[ = k \gamma (1 - \gamma^{-1}) x' - k \gamma (1 - \gamma^{-1}) t' \]
\[ = k' x' - \omega' t' \]

\[ E_y' = \gamma (E_0 - v E_0) \cos (k \gamma (1 - \gamma^{-1}) x' - k \gamma (1 - \gamma^{-1}) t') \]
\[ B_z' = \gamma (E_0 - v E_0) \cos (k \gamma (1 - \gamma^{-1}) x' - k \gamma (1 - \gamma^{-1}) t') \]

What is the wavelength in the new frame?

\[ \lambda' = \frac{2\pi}{k'} = \frac{2\pi}{k \gamma (1 - \gamma^{-1})} = \frac{\lambda}{\gamma (1 - \gamma^{-1})} = \frac{\lambda}{1 - \gamma^{-1}} \cdot \frac{\sqrt{1 - \gamma^{-2}}}{\gamma} = \lambda \sqrt{\frac{1 + \gamma^{-2}}{1 - \gamma^{-2}}} \]

Redshift!

What is the ratio of energy density in the original frame to energy density in the final frame?

\[ \frac{U}{U'} = \gamma^2 (1 - \gamma^{-2}) = \frac{1 - \gamma^{-2}}{1 + \gamma^{-2}} \]

\[ U \propto E_y B_z \Rightarrow \frac{U'}{U} = \gamma^2 (1 - \gamma^{-2}) = \frac{1 - \gamma^{-2}}{1 + \gamma^{-2}} \]
Example

Electric field of a moving charge $q$

Consider a charge moving in the $\hat{x}$ direction with velocity $v \hat{x}$. Consider a distance $r$ away from the charge at an angle $\Theta$ w. r. t. $\vec{v}$.

What is the $|\vec{E}|$ as a function of $\Theta$?

In the rest frame of the charge,

$$|\vec{E}'| = \sqrt{E'_x^2 + E'_z^2}$$

Assume charge is at $x = y = z = 0$. Focus on $y = 0$ plane.
\[ E'_x = \frac{q}{4\pi} \frac{x'}{r'^2} \quad E'_z = \frac{q}{4\pi} \frac{z'}{r'^2} \]

As usual, assume \((t', x') = 0\) coincides with \((t, x) = 0\).

\[ t' = t \gamma - \gamma v x \quad z' = z \]

\[ x' = x \gamma - \gamma v t \]

\[ r'^2 = (x')^2 + (z')^2 = (x \gamma - \gamma v t)^2 + z^2 \]

Since the charge is at the origin at \(t = 0\) and we want the \((x, z)\) spatial geometry when the particle is momentarily at the origin, set \(t = 0\).

\[ r' = \sqrt{(x \gamma)^2 + z^2} = \gamma \sqrt{x^2 + \frac{z^2}{\gamma^2}} = \gamma \sqrt{x^2 + z^2 (1 - v^2)} \]

\[ E_x = E'_x \quad \Rightarrow \quad E_x = \frac{q}{4\pi} \frac{x \gamma}{r'^2} = \frac{q}{4\pi} \frac{1}{\gamma^2 \left(1 - \sin^2 \theta \right)} \frac{x \gamma}{r \sqrt{1 - v^2 \sin^2 \theta}} \]

\[ E_z = \gamma (E'_z - \gamma \beta \gamma) \quad E_z = \frac{q}{4\pi} \frac{z}{r'^2} \gamma = \frac{q}{4\pi} \frac{1}{r^2 \gamma^3} \frac{\gamma \sin \theta}{(1 - \gamma^2 \sin^2 \theta)^{3/2}} \]
\[ |\mathbf{E}| = \sqrt{E_x^2 + E_z^2} \]

\[ = \frac{\frac{q}{4\pi}\frac{l}{r^2}}{\frac{1}{\varepsilon_0} \left(1 - \frac{v^2}{c^2} \sin^2 \theta \right)^{3/2}} = \frac{\frac{q}{4\pi\varepsilon_0 r^2}}{\left[1 - \frac{v^2}{c^2} \sin^2 \theta \right]^{3/2}} \]

This matches equation 12 of page 184 of Purcell.