Lec 31: Begin Relativistic Transformation of $\vec{E}$ and $\vec{B}$

last time: Larmor radiation
today: Begin relativistic transformation of $\vec{E}$ and $\vec{B}$

(Note special relativity from physics 247 - review in Appendix A of Purcell.)
In lecture 27, you learned \( \frac{1}{c^2} \frac{\partial^2 \vec{E}(t, \vec{x})}{\partial t^2} - \nabla^2 \vec{E}(t, \vec{x}) = 0 \) and \( \frac{1}{c^2} \frac{\partial^2 \vec{B}(t, \vec{x})}{\partial t^2} - \nabla^2 \vec{B}(t, \vec{x}) = 0 \) are the wave equations implied by the source-free Maxwell equations.

Source-free means that the waves are propagating in the vacuum.

If vacuum does not have a preferred inertial frame of reference, then any two inertial frames related by constant velocity boost should yield the same equations governing electromagnetic fields.

Recall from page 7 of lecture 15 of physics 247 that a "general" boost in the x direction without imposing that speed of light is

\[
\Lambda(v) = \begin{pmatrix}
  a(v) & \frac{1 - [a(v)]^2}{v a(v)} \\
  -v a(v) & a(v)
\end{pmatrix}
\]

\[
(t', x') = \Lambda(v) (t, x) = \begin{pmatrix}
  a t + \frac{1-a^2}{v a} x \\
  -v a t + a x
\end{pmatrix}
\]

Note we have set \( c = 1 \).

Apply this to the wave solution propagating in the \( \hat{x} \)-direction.
\[ \frac{\partial^2 E'(x')}{\partial t'^2} - \frac{\partial^2 E'(x')}{\partial x'^2} = 0 \quad \Rightarrow \quad \frac{\partial^2 E'(t', x', 0, 0)}{\partial t'^2} - \frac{\partial^2 E'(t', x', 0, 0)}{\partial x'^2} = 0 \]

A solution labeled "1" propagating in the \( \hat{x} \)-direction described in the primed frame.

Using the \( \otimes \) argument of the previous page, there must be a corresponding solution \( \hat{E}_1 \) described in the unprimed frame such that \( \frac{\partial^2 \hat{E}_1}{\partial t^2} - \frac{\partial^2 \hat{E}_1}{\partial x^2} = 0 \)

There cannot be any other source-free equation governing \( \hat{E}_1 \) according to \( \otimes \).

\[ \text{Reexpress} \quad \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \hat{E}_1 = 0 \quad \text{in terms of} \quad \left\{ \frac{\partial}{\partial t'}, \frac{\partial}{\partial x'} \right\} \]

using chain rule:

\[ \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} \frac{\partial}{\partial t'} + \frac{\partial}{\partial x'} \frac{\partial}{\partial x'} \]

\[ \frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \frac{\partial}{\partial t'} + \frac{\partial}{\partial x'} \frac{\partial}{\partial x'} \]

\[ \frac{\partial^2}{\partial t^2} = \left( \frac{\partial}{\partial t'} \frac{\partial}{\partial t'} + \frac{\partial}{\partial x'} \frac{\partial}{\partial x'} \right) \left( \frac{\partial}{\partial t'} \frac{\partial}{\partial t'} + \frac{\partial}{\partial x'} \frac{\partial}{\partial x'} \right) \]

\[ \frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x'} \frac{\partial}{\partial t'} + \frac{\partial}{\partial x'} \frac{\partial}{\partial x'} \]

\[ \frac{\partial t'}{\partial t} = a \]

\[ \frac{\partial x'}{\partial t} = \frac{1 - a^2}{v^2} \]

\[ \frac{\partial x'}{\partial x} = v^2 \]

\[ \frac{\partial t'}{\partial x} = -av \]

\[ \frac{\partial x'}{\partial x} = a \]
You will find in HW that this implies

\[ 2a^2 v + 2 \frac{(1-a^2)}{v} = 0 \]

\[ 1-a^2 = -v^2 a^2 \]

\[ a = \frac{1}{\sqrt{1-v^2}} \]

This is (not surprisingly) the same factor we found in physics 245 assuming that the propagation speed is the same in all frames of reference.

Hence, Maxwell equations chooses Lorentz transformations as the one to be used when boosting to different inertial frames of reference.

Now that we have established the preference for Lorentz transformations, we can find how fields transform as follows.

**Step 1**

Rewrite Maxwell equations as

\[ F^{\mu\nu} = \sum_{\mu=0}^{\infty} \frac{\varepsilon}{\varepsilon - \omega^2} F^{\mu\nu} = \eta^{\gamma \lambda} \] for \( \nu \in \{0, 1, 2, 3\} \) where

\[ F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \]

\[ \eta^{\gamma \lambda} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ \eta = \begin{pmatrix} \rho & \frac{x}{J_0} \lambda \\ \frac{x}{J_0} \lambda & \frac{J_y}{J_0} \lambda \\ \frac{x}{J_0} \lambda & \frac{J_y}{J_0} \lambda \\ \frac{x}{J_0} \lambda & \frac{J_y}{J_0} \lambda \end{pmatrix} \]

set \( \mu_0 = \varepsilon_0 = 1 \) \( \leftrightarrow \) \( c = 1 \)

Restore at end using dimensional analysis.
Check 0th component:

\[
\sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu} F^{\mu\sigma} = j^0 \iff \nabla \cdot \vec{E} = \rho
\]

LHS:

\[
\sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu} F^{\mu\sigma} = \frac{\partial}{\partial x^0} E^0 + \frac{\partial}{\partial x^1} E^1 + \frac{\partial}{\partial x^2} F^{20} + \frac{\partial}{\partial x^3} F^{30}
\]

\[
= \frac{\partial}{\partial x} E_x + \frac{\partial}{\partial y} E_y + \frac{\partial}{\partial z} E_z = \nabla \cdot \vec{E}
\]

RHS:

\[
\rho
\]

(\text{In SI units, } \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0})

You will verify in your HW one of the remaining components

\text{Step 1:}

Define \( A^\mu = \begin{pmatrix} \phi \\ \vec{A} \end{pmatrix} \) where \( \vec{B} = \nabla \times \vec{A} \) and \( \vec{E} = -\frac{\partial}{\partial t} \vec{A} - \nabla \phi \)

In lecture 29, you learned that we can rewrite Maxwell eqs. of step 1 as

\[
(\frac{\partial^2}{\partial t^2} - \nabla^2) A^\mu - \sum_{\mu=0}^{3} \eta^{\mu\nu} \frac{\partial}{\partial x^\nu} \sum_{\alpha=0}^{3} \frac{\partial}{\partial x^\alpha} A^\alpha = j^\mu
\]

\[
\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\]
proof

Consider the 0 component:

\[
\frac{\partial^2}{\partial t^2} A^0 - \nabla^2 A^0 - \frac{\partial}{\partial t} \left( \frac{\partial}{\partial x^i} A_i^0 + \nabla \cdot \vec{A} \right) = \rho - \frac{\rho}{\varepsilon_0} \vec{J}
\]

\text{Step 3}

You will verify your HW that

\[
P^\mu_{\nu} = \sum_{\alpha = 0}^{3} \eta_{\mu\alpha} \frac{\partial}{\partial x^\alpha} A^\nu - \sum_{\rho = 0}^{3} \eta_{\nu\rho} \frac{\partial}{\partial x^\rho} A^\mu
\]

E.g.

\[
P^0_1 = \sum_{\alpha = 0}^{3} \eta_{0\alpha} \frac{\partial}{\partial x^\alpha} A^1 - \sum_{\rho = 0}^{3} \eta_{1\rho} \frac{\partial}{\partial x^\rho} A^0 = \frac{\partial}{\partial t} A_x + \frac{\partial}{\partial \phi} \phi = -E_x
\]

\text{Step 4}

Note that \( f^\mu(x) \) behavior under Lorentz transformations is the same as that of a 4-vector.

\text{proof}

\( d^3x \rho = \text{charge} = \text{Lorentz invariant} \)

\( d^3x \) suffers length contraction

\[
\text{i.e. } d^3x \rightarrow \frac{d^3x}{\gamma}
\]

\[
\therefore \rho \rightarrow \rho \gamma \text{ under a boost}
\]

Note \( dt \rightarrow \gamma dt \) due to time dilation.
\[ i. \quad \phi \text{ transforms like } \partial t. \]

Similarly, \( J_x \, dA_\perp \, dt = \text{Lorentz invariant} \)

\[ \square \text{: How does } d\mathbf{x} \, dA_\perp \, dt \text{ transform under a boost?} \]

\[ \square \text{: Why does this imply that } J_x \text{ transform as } d\mathbf{x}? \]

**Step 5**

Suppose \( V^\mu \) transforms like a 4-vector.

\[ \sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu} V^\mu \text{ is Lorentz invariant} \]

**Proof**

\[ V^\mu \rightarrow \sum_{\nu=0}^{3} \bigwedge^{\nu} \, v \left( \frac{\partial}{\partial x} \right) \, V^\nu \]

\[ \frac{\partial}{\partial x^\mu} \rightarrow \frac{\partial}{\partial x'^\mu} = \sum_{\lambda=0}^{3} \frac{\partial x^\lambda}{\partial x'^\mu} \frac{\partial}{\partial x^\lambda} \]

\[ (dx')^\mu = \sum_{\nu=0}^{3} \bigwedge^{\nu} \, v \left( \frac{\partial}{\partial x} \right) \, dx^\nu \rightarrow \sum_{\mu=0}^{3} \bigwedge^{\mu} \, v \left( \frac{\partial}{\partial x} \right) \, (dx')^\mu = dx^\lambda \]

\[ \Rightarrow \quad \frac{\partial x^\lambda}{\partial x'^\mu} = \bigwedge^{\lambda} \, v \left( \frac{\partial}{\partial x} \right) \quad \Rightarrow \quad \frac{\partial}{\partial x'^\mu} = \sum_{\lambda=0}^{3} \bigwedge^{\lambda} \, v \left( \frac{\partial}{\partial x} \right) \, \frac{\partial}{\partial x^\lambda} \]
\[
\sum_{\mu=0}^{3} \frac{\partial}{\partial x^{\mu}} V^{\mu} \rightarrow \sum_{\mu=0}^{3} \sum_{\lambda=0}^{3} \Lambda^{\mu}_{\nu} (-i\omega) \frac{\partial}{\partial x^{\lambda}} \sum_{\nu=0}^{3} \Lambda^{\nu}_{\nu} (i\omega) V^{\nu}
\]

Note \[\sum_{\mu=0}^{3} \Lambda^{\mu}_{\nu} (-i\omega) \Lambda^{\nu}_{\nu} (i\omega) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\]

\[\therefore \sum_{\mu=0}^{3} \frac{\partial}{\partial x^{\mu}} V^{\mu} \rightarrow \sum_{\lambda=0}^{3} \frac{\partial}{\partial x^{\lambda}} V^{\lambda}\]

**Step 6**

\[\sum_{\mu=0}^{3} \eta^{\mu\nu} \frac{\partial}{\partial x^{\nu}} \text{ transforms as a 4-vector (HW)}\]

**Step 7**

\[\therefore A^{\nu} \text{ transforms as a 4-vector by reexamining Step 2}\]

**Step 8**

Examining step 3 \[\Rightarrow F^{\mu\nu} \rightarrow \sum_{\alpha, \beta=0}^{3} \Lambda^{\mu}_{\alpha}(i\omega) \Lambda^{\nu}_{\beta} (i\omega) F^{\alpha\beta}\]

giving the Lorentz transformation rules for \(\vec{E}\) and \(\vec{B}\).