Physics 248 Lecture 2

Topics:
- Electric fields - discrete and continuous charge distributions
- Electric flux
- Gauss's Law

Last time, we discussed the electrostatic interactions of point charges.

Coulomb's Law:
\[
\vec{F}_{on 2 by 1} = - \vec{F}_{on 1 by 2} = \frac{k q_1 q_2 \hat{r}_{21}}{r_{21}^2} = \frac{k q_1 q_2 (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}
\]

In general, force on \( q \) at \( \vec{r} \) due to \( \{ q_i \} \) at \( \{ \vec{r}_i \} \)

\[
\vec{F}_{on q} = \sum_i \frac{k q q_i \hat{r}_{q_i}}{r_{q_i}^2} = \sum_i \frac{k q q_i (\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^3}
\]

SI:
\[
k = 8.988 \times 10^9 \text{ N m}^2/\text{C}^2 = \frac{1}{4\pi\varepsilon_0} \quad \varepsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2
\]
From this, define the electric field \( \vec{E}(\vec{r}) \) due to charges \( q_i \) at \( \vec{r}_i \):

\[
\vec{E}(\vec{r}) = \sum_i \frac{kq_i (\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^3}
\]

Hence, in this field, the electrostatic force on a charge \( q \) at \( \vec{r} \) is

\[
\vec{F}_{\text{on } q} = q \vec{E}(\vec{r})
\]

Visualize the field as electric field lines:

- curves whose tangent lies in the direction of the field at all points
- start on positive charges
- end on negative charges
- do not start or end in empty space
- never cross other field lines
So far, we've considered only point charges.

Turn now to continuous charge distributions.

In some large-scale system (containing many atoms) compared to atomic size.

Describe charges in a volume by the charge density

\[ \rho(\mathbf{r}) = \text{charge/volume} \]

\[ \rho(\mathbf{r}) \, dV = \text{charge contained in } dV \]

Now when we calculate the electric field, instead of summing over discrete charges, we integrate:

\[ \mathbf{E}(\mathbf{r}) = \int k \frac{\rho(\mathbf{r}') \, (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, dV' \]

Charge density \( \rho(\mathbf{r}') \)

\( \mathbf{r}' \): source point

\( \mathbf{r} \): field point or observation point

\( dV' = dV(\mathbf{r}') \): infinitesimals volume element
This was the case for a volume charge density. One can also consider situations in which the charge is distributed over a surface or a line.

\[ \mathbf{\nabla} (\nabla') = \text{charge/area} \quad \text{surface charge density} \]

\[ \lambda (\nabla') = \text{charge/line} \quad \text{linear charge density} \]

Can compute electric field in these cases as well:

\[
E(\mathbf{r}) = \int k \frac{\sigma(\nabla') (\mathbf{r} - \nabla')}{|\mathbf{r} - \nabla'|^3} \, da'
\]

\[
\mathbf{E}(\mathbf{r}) = \int k \frac{\lambda(\nabla') (\mathbf{r} - \nabla')}{|\mathbf{r} - \nabla'|^3} \, dl'
\]

In principle, can compute electrostatic field using brute force integration for any distribution. Drawback: complicated integral of a vector function.
Flux & Gauss's Law

Imagine an arbitrarily sized surface (balloon) in a region with a nonzero electric field.

Surface divided into infinitesimal area element \( \mathbf{d}\mathbf{a} \) (magnitude \( da \), direction \( \mathbf{n} \) to surface).

Flux through the surface:

\[
\text{flux} = \int \mathbf{E} \cdot \mathbf{d}\mathbf{a}
\]

\( \int_{\text{surface}} \)

Can consider flux through open surface or closed surface.

\[
\int_{S} \mathbf{E} \cdot \mathbf{d}\mathbf{a}
\]

\( \int_{S} \)

Note for closed surfaces → flux vanishes unless charges are in the volume of the region bounded by the surface \( S \).

(reasoning → charges source field lines)

(whitewashed section)

(whitewashed section)
This fact will lead us to Gauss's Law, one of the four Maxwell's Equations, which relates the flux of the electric field to the charge enclosed.

To see this → consider a point charge $q$

$$\mathbf{E}(\mathbf{r}) = \frac{kq}{r^2}$$

Consider the flux of this field through a sphere of radius $r$ centered on the charge

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{kq}{r^2} 4\pi r^2 = 4\pi kq = \frac{q}{\varepsilon_0}$$

The flux is independent of the size of the sphere.

We can generalize this argument for surface of arbitrary shape.

This is the integral form of Gauss's Law:

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{q_{\text{enclosed}}}{\varepsilon_0}$$

$q_{\text{enclosed}} =$ charge enclosed in the volume bounded by surface $S$. 
or rewriting \[ \oint E \cdot d\mathbf{a} = \frac{q_{\text{enclosed}}}{\varepsilon_0} = \int \rho \, dV \]

Gauss's law is an equivalent statement of Coulomb's law.

It directly follows from the inverse square nature of Coulomb's law.

Gauss's law, in certain cases, can be used as a tool to find \( \vec{E} \).

Normally \( \oint \vec{E} \cdot d\mathbf{a} \) is a difficult integral to do (for most surfaces).

What if one can find a "Gaussian surface" in which:

- \( \vec{E}(\vec{r}) \) constant at all points on the surface
- Direction of \( \vec{E} \) always perpendicular to the surface at all points (i.e. \( \vec{E} \parallel d\mathbf{a} \) always)

Then \[ \oint \vec{E} \cdot d\mathbf{a} = \int \vec{E} \cdot d\mathbf{a} = \vec{E} \cdot \mathbf{A} \]

↑ area of surface
Example: point charge $q$.

We know that the $\vec{E}$ of $q$ is radially outward and only depends on the distance $r$.

Hence, over spheres of constant radius, we have $\vec{E}(\vec{r})$ fixed in magnitude and always parallel to $d\vec{a}$:

$$\oint \vec{E} \cdot d\vec{a} = EA = \frac{q \text{ enclosed}}{\varepsilon_0} = \frac{q}{\varepsilon_0}$$

$$\Rightarrow \quad E = \frac{q}{\varepsilon_0} \frac{1}{A} = \frac{q}{4\pi \varepsilon_0 r^2} = \frac{kq}{r^2}$$

$$\Rightarrow \quad \vec{E}(\vec{r}) = \frac{kq}{r^2} \hat{\vec{r}} \quad \text{as expected.}$$

Under what circumstances can we use the brick?

Need a high degree of symmetry: spherical, cylindrical, planar.
Gauss's law examples:

Spherical symmetry

Charge distributions that only depend on the radial coordinate \( r \)
(not the angular coordinates \( \Theta, \Phi \))

Gaussian surface: sphere of radius \( r \)

\[
\oint E \cdot d\mathbf{a} = E \oint d\mathbf{a} = Ey\pi r^2 = q_{\text{enclosed}}/\varepsilon_0
\]

\[
\Rightarrow \quad E(\hat{r}) = \frac{q_{\text{enclosed}}}{4\pi \varepsilon_0 r^2}
\]

Example: uniform volume charge density in sphere of radius \( a \)

Total charge on sphere:

\[
q = \int \rho \, dV = \rho \frac{4}{3} \pi a^3
\]

For \( r > a \):

\[
q_{\text{enclosed}} = q \quad \Rightarrow \quad E(\hat{r})_{r > a} = \frac{q}{4\pi \varepsilon_0 r^2} \quad \hat{r} = \frac{pa^3}{3\varepsilon_0 r^2}
\]

For \( r < a \):

\[
q_{\text{enclosed}} = \int \rho \, dV = \rho \frac{4}{3} \pi r^3 \quad \Rightarrow \quad E(\hat{r})_{r < a} = \frac{\rho r^2}{3\varepsilon_0} \quad \hat{r} = \frac{qr}{4\pi \varepsilon_0 a^3} \quad \hat{r}
\]
Gauss's law examples: cylindrical symmetry

Linear/cylindrical systems with infinitely long charges along main axis
e.g. line charge, cylinder

only depends on radial coordinate \( r \) in cylindrical coordinates
(not \( \phi \) or \( z \))

Gaussian surface: cylinders of radius \( r \), length \( l \)

\[
\oint E \cdot d\mathbf{a} = E \oint d\mathbf{a} = E \int 2\pi r \, dl = \frac{q_{\text{enclosed}}}{\varepsilon_0}
\]

\[
E(r) = \frac{q_{\text{enclosed}}}{2\pi r l \varepsilon_0} \ \ \ \text{cylindrical coord} \ \ \hat{r}
\]

Example: infinitely long line charge, linear charge density \( \lambda \)

\[
q_{\text{enclosed}} = \lambda l \ \ \Rightarrow \ E(r) = \frac{\lambda l}{2\pi r l \varepsilon_0} \ \ \hat{r} = \frac{\lambda}{2\pi r \varepsilon_0} \ \ \hat{r}
\]