Lec 16: Magnetic Field and Biot-Savart Law

Last time: intro to semiconductor circuits

Thus far, \( \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \) and \( \vec{F} = q \vec{E} \)

one of Maxwell equations

will add to today

Today: Begin magnetic fields

Reading: Chapter 6 of Purcell (will return to chap 5 later)
What is the force on a moving charge?

\[ \vec{F} = q \vec{E} + q \vec{v} \times \vec{B} \]

experimentally established

where \( \vec{B} \) is a new kind of field called magnetic field.

The magnetic field is produced by currents. Biot, Savart, and Ampere deduced the following from experiments involving steady currents:

\[ \vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{r}}{r^2} \, dl \]

where \( r \) is a function of \( l \)

\[ \mu_0 = 4\pi \times 10^{-7} \frac{N}{A^2} \approx 1.26 \times 10^{-6} \frac{N}{A^2} \]

SI units of \( \vec{B} \): Tesla = \( \frac{N}{mA} \)
comments:
1) Magnetic force is on a moving charge
2) Magnetic force is \( \vec{F} \) to \( \vec{v} \) \( \implies \) No work on charge
3) Magnetic field is due to currents, which can exist even when the net charge is zero
4) Although an analog of Coulomb's law for magnetostatics, the current is different in that an isolated current element does not exist.
5) As in the relationship (differential) Gauss's law \( \leftrightarrow \) Coulomb's law, the Biot-Savart law has a differential form as we will see.
6) The Biot-Savart law is for steady currents
7) Surface current
   \[
   \vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \hat{r}}{r^2} \, da
   \]
   \[
   \vec{J} \, dl \, dl = \vec{I} \, dl
   \]
   Volume current
   \[
   \vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \hat{r}}{r^2} \, dV
   \]
   \[
   \vec{J} \, da \, dl = \vec{I} \, dl
   \]
Example

At $t=0$, a charged particle of mass $m$ and charge $q$ has velocity $\mathbf{v}$ at position $(0,0,0)$. What is the position of the particle at all other times if there is a uniform magnetic field $\mathbf{B} = B_0 \hat{z}$?

Ans.

\[ \mathbf{F} = q \mathbf{v} \times \mathbf{B} = q |\mathbf{v}| |\mathbf{B}| \hat{\theta} \]

where $\hat{\theta}$ is perpendicular to $\mathbf{v}$

\[ \mathbf{F} = q \mathbf{v} B_0 \hat{f} \]

\[ \frac{m \mathbf{v}^2}{r} = q \mathbf{v} B_0 \]

\[ r = \frac{m \mathbf{v}}{q B_0} \]
From physics 247, we know
\[ \vec{x} = (r \sin(\omega t), r[\cos(\omega t) - 1], 0) \]

where \( r \omega = \nu \Rightarrow \omega = \frac{\nu}{(\frac{m \nu}{g B_0})} = \frac{g B_0}{m} \)

**Example**

Find the magnetic field a distance \( r \) above a straight wire carrying a steady current \( I \).

**Ans**

\[ D = \sqrt{r^2 + l^2} \]
\[ \sin \phi = \frac{r}{D} \]

\[ B = \frac{\mu_0 I}{4\pi} \int \frac{\vec{T} \times \vec{D}}{D^2} \, dl = \frac{\mu_0 I}{4\pi} \left( \int_{-\infty}^{\infty} \frac{dl}{\sqrt{r^2 + l^2}} + \int_{-\pi/2}^{\pi/2} \frac{d\theta \sec^2 \theta}{\sqrt{r^2 + l^2}} \right) = \frac{\mu_0 I}{4\pi} \left( \frac{\pi}{2} \int_{-\infty}^{\infty} \frac{dl}{\sqrt{1 + (\frac{L}{r})^2}} \right) \]

\[ \frac{d}{dl} = \tan \theta \]
\[ dl = r \sec^2 \theta \, d\theta \]
Example (Section 6.1)

Find the force/length between two parallel wires carrying currents $I_1$ and $I_2$ in the same direction and distance $r$ apart.

\[ \vec{F} = \frac{\mu_0}{4\pi} \frac{I_1 I_2}{r} \hat{\phi} \]

\[ B = \frac{\mu_0 I_2}{2\pi r} \]

\[ \vec{F}_{\text{mutual}} = -\frac{\mu_0 I_1 I_2}{4\pi r} \hat{\phi} \]
Since

\[ D = \sqrt{r^2 + l^2} \]

produces a \( \hat{B} \) pointing in the \( -\hat{z} \) direction instead of the \( \hat{z} \) direction of the \( \hat{B} \) field due to \( I_2 \)

\[ \hat{F}_2 = (\text{length}) \left( \hat{r} \times \left( \nu_2 \times \hat{L} \right) \right) \times (-\hat{B} \hat{z}) \]

\[ \Rightarrow \frac{\hat{F}_{\alpha}}{\text{length}} = \frac{\mu_0 I_1 I_2}{2\pi r} \hat{y} \]

\[ \Rightarrow \text{wires attract} \]
Consider \( \nabla \times \vec{B} = \mu_0 \vec{J} \) and \( \nabla \cdot \vec{B} = 0 \).

We want to show that these imply the Biot-Savart law.

Fact:
The most general vector function \( \vec{B} \) can be written as
\[
\vec{B} = \nabla \times \hat{A} + \nabla C
\]
where \( \hat{A} \) is called the vector potential.

Since \( \nabla \cdot \vec{B} = 0 \) and \( \nabla \cdot (\nabla \times \hat{A}) = 0 \leftarrow \text{(HW)} \),
\[
\nabla \cdot \nabla C = \nabla^2 C = 0
\]
with the vanishing boundary conditions, \( C = 0 \) uniquely.

\[
\therefore \quad \vec{B} = \nabla \times \hat{A} \quad \text{(section 6.3)}
\]

\[
\nabla \times \vec{B} = \nabla \times (\nabla \times \hat{A}) = \nabla (\nabla \cdot \hat{A}) - \nabla^2 \hat{A} \leftarrow \text{(HW)}
\]

\[
\therefore \quad -\nabla^2 \hat{A} = \mu_0 \vec{J}
\]

Since this form is the same as Poisson's equation,
we can write down the solution:

\[
\hat{A}(\mathbf{x}_1) = \frac{\mu_0}{4\pi} \int dV_2 \frac{\hat{\mathbf{J}}(\mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|}
\]

\[
d\hat{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{\hat{\mathbf{J}}(\mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|} dV_2
\]

\[
d\hat{\mathbf{B}}(\mathbf{x}_1) = \nabla \times (d\hat{A}) = \frac{\mu_0}{4\pi} \begin{vmatrix}
\mathbf{x} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\
\frac{2}{\theta x_1} & \frac{2}{\theta y_1} & \frac{2}{\theta z_1} \\
\frac{J_1}{r} & \frac{J_2}{r} & \frac{J_3}{r}
\end{vmatrix}
\]

\[
r = |\mathbf{x}_1 - \mathbf{x}_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}
\]

\[
\frac{2}{\theta y_1} \frac{1}{r} = \frac{1}{r^3} \left( -\frac{1}{2} \right) 2 (y_1 - y_2) = \frac{(y_1 - y_2)}{r^3}
\]

\[
= \frac{\mu_0}{4\pi} \left[ \mathbf{x} \left( -\mathbf{J}_3 \frac{y_1 - y_2}{r^3} + \mathbf{J}_2 \frac{z_1 - z_2}{r^3} \right) + \cdots \right]
\]

Q: Why?
\[ \hat{J} \times \hat{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{J_1}{r} & \frac{J_2}{r} & \frac{J_3}{r} \\ \frac{x_1 - x_2}{r} & \frac{y_1 - y_2}{r} & \frac{z_1 - z_2}{r} \end{vmatrix} \]

\[ = \hat{x} \left( \frac{J_2}{r} \frac{z_1 - z_2}{r} - \frac{J_3}{r} \frac{y_1 - y_2}{r} \right) + \cdots \]

\[ \Rightarrow \quad d\vec{B}(\hat{r}_1) = \frac{\mu_0}{4\pi} \frac{\hat{J} \times \hat{r}}{r^2} \, dV_2 \]

\[ \vec{B}(\hat{r}) = \frac{\mu_0}{4\pi} \int dV_2 \frac{\hat{J}(\hat{r}_2) \times \hat{r}}{r^2} \quad \hat{r} \equiv \hat{x}_1 - \hat{x}_2 \]

\[ \text{Biot-Savart law} \]