1.9 A spherical volume of radius \( a \) is filled with charge of uniform density \( \rho \). We want to know the potential energy \( U \) of this sphere of charge, that is, the work done in assembling it. Calculate it by building the sphere up layer by layer, making use of the fact that the field outside a spherical distribution of charge is the same as if all the charge were at the center. Express the result in terms of the total charge \( Q \) in the sphere.

\[
U = \int_0^a \left( \frac{4\pi r^2}{3\varepsilon_0} \right) \rho \, dr = \frac{4\pi}{3\varepsilon_0} \frac{1}{5} a^5
\]

\[
\rho \frac{4\pi}{3} a^3 = Q \\
\rho = \frac{3}{4\pi} \frac{Q}{a^3} \Rightarrow \quad U = \frac{3}{4\pi} \frac{Q}{\varepsilon_0} \frac{a^3}{5}
\]

1.19 An infinite plane has a uniform surface charge distribution \( \sigma \) on its surface. Adjacent to it is an infinite parallel layer of charge of thickness \( d \) and uniform volume charge density \( \rho \). All charges are fixed. Find \( E \) everywhere.

\[
\vec{E} = \vec{E}_1 + \vec{E}_2
\]

\[
\vec{E}_1 = \left\{ \begin{array}{ll}
\frac{\sigma}{2\varepsilon_0} \hat{x} & x > 0 \\
-\frac{\sigma}{2\varepsilon_0} \hat{x} & x < 0
\end{array} \right.
\]

For \( \vec{E}_2 = \vec{E}_2 \hat{x} \), use pill box centered about \( x = \frac{d}{2} \):

\[
x = \frac{d}{2}
\]
\[ P_{2A} = \begin{cases} \frac{Ad}{2\varepsilon_0} & \text{for } d < x \\ \frac{A(x-d)^2}{2\varepsilon_0} & \text{for } \frac{d}{2} < x < d \\ -\frac{A(d-x)^2}{2\varepsilon_0} & \text{for } 0 < x < \frac{d}{2} \\ -\frac{A}{2\varepsilon_0} & \text{for } x < 0 \end{cases} \]

\[ P_{2} = \begin{cases} \frac{d\rho}{2\varepsilon_0} & \text{for } d < x \\ \frac{(x-d)^2\rho}{2\varepsilon_0} & \text{for } \frac{d}{2} < x < d \\ -\frac{(d-x)^2\rho}{2\varepsilon_0} & \text{for } 0 < x < \frac{d}{2} \\ -\frac{\rho}{2\varepsilon_0} & \text{for } x < 0 \end{cases} \]

\[ \frac{1}{P} = \hat{x} \left\{ \begin{array}{l} \frac{5}{2\varepsilon_0} + \frac{pd}{2\varepsilon_0} & \text{for } d < x \\ \frac{5}{2\varepsilon_0} + \frac{(x-d)^2}{2\varepsilon_0} & \text{for } 0 < x < d \\ -\frac{5}{2\varepsilon_0} - \frac{pd}{2\varepsilon_0} & \text{for } x < 0 \end{array} \right. \]

1.21 The neutral hydrogen atom in its normal state behaves in some respects like an electric charge distribution which consists of a point charge of magnitude $e$ surrounded by a distribution of negative charge whose density is given by $-\rho(r) = Ce^{-2r/a_0}$. Here $a_0$ is the Bohr radius, $0.53 \times 10^{-8}$ cm, and $C$ is a constant with the value required to make the total amount of negative charge exactly $e$. What is the net electric charge inside a sphere of radius $a_0$? What is the electric field strength at this distance from the nucleus?

\[ Q_{\text{cloud}} = -4\pi \int_0^{a_0} r^2 C e^{-2r/a_0} dr = -4\pi C a_0 \frac{(e^2-5)}{4e^2} \]

\[ = -4\pi \int_0^{\infty} r^2 C e^{-2r/a_0} dr = -q_p \text{ where } q_p = \text{charge of proton} \]

\[ C = \frac{q_p}{a_0^3 \pi} \]

\[ Q_{\text{cloud}} = -4\pi \frac{q_p}{a_0^3 \pi} \frac{(e^2-5)}{4e^2} = -q_p \left(1 - \frac{5}{e^2}\right) \approx -0.3 q_p \]

Net charge

\[ Q_{\text{net}} = 0.7 q_p \]

\[ \Rightarrow \left| \frac{E}{|E|} \right| \frac{0.7 q_p}{4\pi \varepsilon_0 a_0^2} \approx 3.6 \times 10^{11} \text{ N} \]
4. A hemispherical body of radius $R$ is centered on the origin with its base lying in the $x-y$ plane and the surface in the region $z > 0$. Calculate the flux

$$\Phi = \int_{\text{hemisphere}} \hat{d}A \cdot \vec{U}$$

through the hemispherical surface of the vector field (function) $\vec{U} = \hat{z}Cz^2/R^2$ where $C$ is a constant. (The $d\hat{A}$ is directed away from the origin as usual.)

$$d\hat{A} = \hat{r} \, d\theta \, d\phi \, r^2 \sin\theta$$
$$d\hat{A} \cdot \vec{U} = \hat{r} \cdot \hat{z} \cdot \frac{Cz^2}{r^2} \sin\theta \, d\theta \, d\phi$$
$$z = R \cos\theta$$

$$\Rightarrow d\hat{A} \cdot \vec{U} = CR^2 \cos^3\theta \sin\theta \, d\theta \, d\phi$$

$$= CR^2 \cos^3\theta \sin\theta \, d\theta \, d\phi$$

$$\therefore \int d\hat{A} \cdot \vec{U} = CR^2 2\pi \int_0^{\pi/2} \cos^3\theta \sin\theta \, d\theta = \frac{CR^2 \pi}{2}$$

\[\square\]

5. \[P2.1\]

$$\hat{E} = 6x\hat{x} + 3(2x^2 - y^2)\hat{y}$$

\begin{align*}
\int \hat{E} \cdot d\vec{R} &= \int_0^{x_1} dx \, 6xy \bigg|_y = 0 + \int_{y_1}^{y_1} dy \, 3(2x^2 - y^2) \bigg|_{x = x_1} \\
&= 3x_1^2 y_1 - y_1^3
\end{align*}

\begin{align*}
\int \hat{E} \cdot d\vec{x} &= \int_0^{y_1} dy \, 8(x^2 - y^2) \bigg|_{x = 0} + \int_0^{x_1} dx \, 6xy \bigg|_{y = y_1} \\
&= -y_1^3 + 3x_1^2 y_1
\end{align*}

The two paths give the same answer as expected.

$$\therefore \Phi = y_1^3 - 3x_1^2 y_1 \quad \therefore \nabla \phi = \begin{bmatrix} \frac{\partial \phi}{\partial x} \hat{x} + \frac{\partial \phi}{\partial y} \hat{y} + \frac{\partial \phi}{\partial z} \hat{z} \end{bmatrix}$$
\[ -\nabla \phi = 6xy \hat{x} - (3y^2 - 3x^2) \hat{y} \]

which matches the original \( \vec{E} \) field as expected.

6. A thin rod extends along the \( z \) axis from \( z = -a \) to \( z = a \), carrying a uniform charge density \( \lambda \).

a) Calculate the potential at all points along the positive \( x \) axis.

b) Calculate the electric field at all points in the \( x - y \) plane.

\[ \phi = \int_{a}^{\infty} \frac{\lambda \, dz}{\sqrt{x^2 + z^2}} \]

\[ = \lambda \ln \left[ \frac{a + \sqrt{x^2 + a^2}}{-a + \sqrt{x^2 + a^2}} \right] \]

\[ \phi = \int_{-a}^{a} \frac{\lambda \, dz}{\sqrt{x^2 + y^2 + z^2}} \]

\[ = \lambda \ln \left[ \frac{a + \sqrt{x^2 + y^2 + a^2}}{-a + \sqrt{x^2 + y^2 + a^2}} \right] \]

\[ \vec{E} = -\nabla \phi = -\left( \frac{\partial \phi}{\partial x} \hat{x} + \frac{\partial \phi}{\partial y} \hat{y} \right) \]

\[ = \frac{2a \lambda \left( x \hat{x} + y \hat{y} \right)}{(x^2 + y^2) \sqrt{a^2 + x^2 + y^2}} \]