Electric Fields and Gauss’s Law

http://www.cco.caltech.edu/~phys1/java/phys1/EField/EField.html

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Electric flux

- Electric flux is a measure of the number of field lines passing through a surface.
- The flux is the product of the magnitude of the electric field and the surface area, \( A \), perpendicular to the field.
- \( \Phi E = EA \)
- Units: \( N \cdot m^2 / C \)

Flux through an arbitrary surface

- A surface element with normal vector \( \mathbf{n} \) and area \( dA \) is characterized by the vector \( d\mathbf{A} = dA \mathbf{n} \).
- The flux through that element is the inner product \( \mathbf{E} \cdot d\mathbf{A} \).
- The flux through a finite surface is the integral over surface elements.

\[
\Phi E = \sum_i \mathbf{E}_i \cdot d\mathbf{A}_i = \int_S \mathbf{E} \cdot d\mathbf{A}
\]
Flux through a closed surface

- Consider a closed surface.
- The surface element vectors are perpendicular to the surface and, by convention, they point outward.
- The contributions to the net outward going flux may be positive, negative, or zero.
- The net flux is proportional to the number of field lines exiting minus the number entering.

Flux of a point charge

- Consider a spherical surface surrounding a point charge.
- At each surface element the electric field has magnitude $E = k_e q / r^2$ and points along the surface normal.
- The net flux is $E(r)A = q / \epsilon_0$.
- $\epsilon_0 \equiv \frac{1}{4\pi k_e}$
- $= \frac{q}{\epsilon_0}$
- $= 8.854 \times 10^{-12} \text{F m}^{-1}$

Arbitrary surface surrounding a charge

- Closed surfaces of various shapes can surround the charge. Only $S_1$ is spherical.
- A little bump or dent in a spherical surface does not change the exiting flux. Many bumps lead to an arbitrary surface.
- The net electric flux is the same through all surfaces.
- The net flux through any closed surface surrounding a point charge $q$ is $q / \epsilon_0$ independent of the shape of the surface.

Closed surface outside a charge

- Consider a closed surface with an arbitrary shape NOT containing a charge.
- Any field line entering the surface leaves at another point.
- The electric flux through a closed surface that surrounds no net charge is zero.
Gauss’s law

* The electric field of multiple charges is the vector sum of electric fields of the individual charges.
* The flux through a surface due to a collection of charges is the scalar sum of the fluxes.
* The net flux through a closed surface of any exterior charge vanishes so the net flux through any closed surface is proportional to the charge enclosed.

\[
\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{A} = \sum q \int_S \mathbf{E}_q \cdot d\mathbf{A} = \sum_{q \text{ enclosed}} \frac{q}{\epsilon_0} = \frac{Q_{\text{enclosed}}}{\epsilon_0}
\]

Field of a uniformly charged sphere

* Pick a spherical surface concentric with and outside a uniform sphere of charge and apply Gauss’s law.
* By symmetry the field is radial.
* We deduce the exterior field is that of a point charge located at the center of the sphere.
* The exterior field of ANY spherically symmetric charge distribution is that of a point charge at the center.

\[
\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{A} = E(r)A = E(r)4\pi r^2 = \frac{Q_{\text{enclosed}}}{\epsilon_0} = \frac{Q}{\epsilon_0} \Rightarrow E(r) = \frac{1}{4\pi \epsilon_0} \frac{Q}{r^2}
\]

Field inside a uniformly charged sphere

* For a radius inside the sphere, only the charge within the Gaussian surface contributes.

\[
\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{A} = E(r)A = E(r)4\pi r^2 = \frac{Q_{\text{enclosed}}}{\epsilon_0} = \frac{Q(r)}{\epsilon_0} \Rightarrow E(r) = \frac{1}{4\pi \epsilon_0} \frac{Q(r)}{r^2}
\]

\[
Q(r) = \frac{4}{3} \pi r^3 \frac{Q}{\frac{4}{3} \pi R^3} = Q \frac{r^3}{R^3}
\]

\[
E(r) = \frac{1}{4\pi \epsilon_0} \frac{Q}{R^3}
\]

Uniformly charged sphere summary

* Inside the sphere, E varies linearly with r

* E \to 0 as r \to 0

* The field outside the sphere is equivalent to that of a point charge located at the center of the sphere.

* The same results apply to gravitation.
**Field of infinite uniformly charged wire**

- Pick a Gaussian surface of cylindrical shape. The field is radial so there is no flux through the top and bottom.
- We recover the result found by direct integration.

\[
\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{A} = E(r)A = E(r)2\pi rl \]

\[
\Phi_E = \frac{\lambda}{\varepsilon_0} \Rightarrow E(r) = \frac{1}{2\pi \varepsilon_0} \frac{\lambda}{r}
\]

**Field of infinite plate of charge**

- Pick a cylinder spanning the surface equally in both directions.
- By symmetry, the field points away from the plate and is constant at fixed distance from the plate.

\[
\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{A} = 2EA \]

\[
\Phi_E = \frac{\varepsilon_0}{\sigma} \Rightarrow E = \frac{\sigma}{2\varepsilon_0}
\]

**Conductors in electrostatic equilibrium**

- Excess charge on or within a conductor quickly rearranges arriving at an equilibrium state in which the excess charge resides only on the surface.
- The electric field vanishes inside the conductor (else charge would flow).
- The electric field at the surface is normal to the surface else charge would flow along the surface. The electric force on the surface charge is balanced by a surface force on the atomic scale.

\[
\mathbf{E} = 0 \text{ inside so no flux exits an arbitrary surface. No charge is found inside.}
\]

**Surface field and surface charge density**

- Apply Gauss’s law to a pill box spanning the surface.
- The interior field vanishes and the exterior field is normal to the surface.
- The only contribution to the flux is from the exterior surface.
- Deduce the surface electric field is proportional to the surface charge density.

\[
\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{A} = EA = \frac{\epsilon_0}{\sigma} \Rightarrow E = \frac{\sigma}{\epsilon_0}
\]

Note: No factor of 1/2 here!
**Example**

- Consider a sphere with charge \(+Q\) surrounded by a shell of charge \(-2Q\).
- Use a Gaussian surface for each region, count the charge inside, and deduce the (radial) electric field.

\[
E_1 = \frac{k_e}{a^2} \frac{Q}{r} \quad \text{(for } r < a) \\
E_2 = \frac{k_e}{r^2} \frac{Q}{r} \quad \text{(for } a < r < b) \\
E_3 = 0 \quad \text{(for } b < r < c) \\
E_4 = -\frac{k_e}{r^2} \frac{Q}{r} \quad \text{(for } r > c)
\]

**Example continued**

- What are the charges on the inner and outer surfaces of the shell?
- The inner charge must be \(-Q\) to balance the charge \(+Q\) inside so that the field within the shell vanish.
- In the absence of net charge on the outer shell, a charge \(+Q\) would reside on its outer surface. If additional charge \(-2Q\) is added to the outer shell, it resides on the outer surface making a net charge \(-Q\) on the outer surface. The total charge of the system is \(-Q\).

**Electric work**

- Consider the work required to move a charge \(+q\) from point A to point B in the radial electric field of a fixed charge \(+Q\). An external force force \(F = -qE\) must be supplied.
- Two paths are shown.
- The differential work in any path segment \(dW = Fdx = -qEdx\) is no vanishing only on the radial segments.
- \(W(A->a) = W(c->d), W(a->b) = W(e->B)\)
- The total work is independent of path. The force field is “conservative.”

**Electric energy**

- The total work \(W\) that I must do to move a charge \(+q\) from A to B in the radial electric field of a fixed charge \(+Q\) is the integral of the differential work \(dW = Fdx = -qEdx\) and follows from the work along a radial path:

\[
W = k_e qQ \left( \frac{1}{r_B} - \frac{1}{r_A} \right)
\]

- By the work energy theorem of mechanics, this work is the change in the potential energy.
- If \(+q\) moves from A to B subject ONLY to the electric field of \(+Q\), the change in its kinetic energy is the negative of this work.
Electric potential

- The potential energy of q is constant on spherical surfaces perpendicular to the electric field of Q.

- The electric potential is the potential energy per unit charge at any point. Any one point may be assigned a value of zero. Only potential differences are of physical significance. $V(B) - V(A) = \frac{U_q(B)}{q} - \frac{U_q(A)}{q}$

- The unit of electric potential is the J C$^{-1}$ = Volt.

- The unit of electric field can be written as N C$^{-1}$ = V m$^{-1}$.

Energy to move a charge in the electric field of a collection of charges

- The electric field of a collection of charges is the vector sum of the electric fields of the individual charges.

- The work by the total electric field is the sum of the work by the individual electric fields.

$$W = \int_A^B \mathbf{E}_{tot} \cdot d\mathbf{x} = \sum_i \int_A^B \mathbf{E}_i \cdot d\mathbf{x} = \sum_i k_c q Q_i \left( \frac{1}{r_{Bi}} - \frac{1}{r_{Ai}} \right)$$