Electric Fields and Gauss’s Law

http://www.cco.caltech.edu/~phys1/java/phys1/EField/EField.html

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The electric force between two point-like charges is given by Coulomb’s Law.

The electric field at a point in space is the total electric force per unit charge on a test charge at that point.

The field may be visualized by field lines.

The electric field of a collection of point charges in the vector sum of the electric fields of the individual charges.

\[ F_{12} = k_e \frac{q_1 q_2}{r_{12}^2} \hat{r}_{12} \]
Electric field of a continuous charge distribution

* Divide the charge distribution into small elements, each of which contains a charge $\Delta q_i$.

* Calculate the electric field due to one of these elements at point $P$.

* Evaluate the total field by summing the contributions of all the charge elements.

$$\vec{E} = k_e \lim_{\Delta q_i \to 0} \sum_i \frac{\Delta q_i}{r_i^2} \hat{r}_i = k_e \int \frac{dq}{r^2} \hat{r}$$
Charge densities

\[ dq(x') = \rho(x') dV' = \rho(x') d^3x' \]
\[ dq(x') = \sigma(x') dA' \]
\[ dq(x') = \lambda(x') dx' \]

- Volume charge density: when a charge is distributed evenly throughout a volume
  - \( \rho \equiv Q / V \) with units C/m\(^3\) “rho”

- Surface charge density: when a charge is distributed evenly over a surface area
  - \( \sigma \equiv Q / A \) with units C/m\(^2\) “sigma”

- Linear charge density: when a charge is distributed along a line
  - \( \lambda \equiv Q / \ell \) with units C/m “lambda”

\[ \mathbf{E}(x) = \sum_i k_e \frac{dq_i}{r_i^2} \hat{r}_i \]
\[ = \sum_i \frac{\rho(x_i) dV_i}{(x - x_i)^2} \frac{x - x_i}{|x - x_i|} \]
\[ \rightarrow \int \frac{\rho(x') d^3x'}{(x - x')^2} \frac{x - x'}{|x - x'|} \]
Problem solving strategies

* Analyzing a group of individual charges:
  * Use the superposition principle, find the fields due to the individual charges at the point of interest and then add them as vectors to find the resultant field.
  * Be careful with the manipulation of vector quantities.

* Analyzing a continuous charge distribution:
  * The vector sums for evaluating the total electric field at some point must be replaced with vector integrals.
  * Divide the charge distribution into infinitesimal pieces, calculate the vector sum by integrating over the entire charge distribution.

* Symmetry:
  * Take advantage of any symmetry to simplify calculations.
Field of a line charge

- To find the field of a uniformly charged rod, pick a coordinate system aligned with the rod and consider the charge $dq$ in a small element of rod.

- Compute the contribution to the field at a point $x$ from an element at $x'$.

\[
\begin{align*}
\mathbf{dE}(x) &= k_e dq \frac{x - x'}{|x - x'|^3} \\
&= k_e \lambda dx' \frac{x - x'}{|x - x'|^3} \\
&= k_e \lambda dx' \frac{x - x'}{((x - x')^2 + (y - y')^2)^{3/2}} \hat{x} \\
&\quad + k_e \lambda dx' \frac{y - y'}{((x - x')^2 + (y - y')^2)^{3/2}} \hat{y}
\end{align*}
\]
Compute the components

- With the help of integral calculus, we can compute the components of the total electric field.

\[ E_y(x) = \int_{a}^{b} k_e \lambda \frac{y}{((x' - x)^2 + y^2)^{3/2}} \, dx' \]

\[ = \frac{k_e \lambda}{y} \frac{x' - x}{\sqrt{(x' - x)^2 + y^2}} \bigg|_a^b \]

\[ = \frac{k_e \lambda}{y} \left[ \frac{b - x}{\sqrt{(b - x)^2 + y^2}} - \frac{a - x}{\sqrt{(a - x)^2 + y^2}} \right] \]

\[ \rightarrow \frac{2k_e \lambda}{y} (a \to -\infty, b \to \infty) \]

- The radial \( y \)-component for an infinitely long line charge is inversely proportional to distance from the line charge.
Field of an infinite uniform line charge

By symmetry, the field of an infinitely long line charge points in the radial direction. The magnitude is proportional to the inverse of radial distance.

\[ E(x) = \frac{2k_e \lambda}{\rho} \hat{\rho} \]

\[ dq = \lambda dx' \]

By symmetry, the field of an infinitely long line charge points in the radial direction. The magnitude is proportional to the inverse of radial distance.
A discrete model

- A JAVA applet is used to find the field of a finite collection of equal charges approximating a uniformly charged rod.

- The field lines flow away from the charges. (A few errant field lines crossing other lines are errors in the calculation.)

- Near the line charge, the field appears as if the collection had infinite extent.

http://www.cco.caltech.edu/~phys1/java/phys1/EFIELD/EFIELD.html
Another example

- To find the field on the axis of a uniformly charged disk, divide the disk into rings of infinitesimal width dr.

- Find the field at the point x due to each ring and then sum the field over rings converting the sum to an integral.

- For disk radius $R \gg x$, the field is that of an “infinite” plane of charge - uniform and normal to the disk.
Motions of charges

• A electron is projected horizontally into a uniform electric field.

• The electron undergoes a downward acceleration. The electron charge is negative, so the acceleration is opposite the direction of the field.

• The motion is parabolic while between the plates.

\[
\begin{align*}
\mathbf{F} &= q\mathbf{E} = -e\mathbf{E} = m_e\mathbf{a} \\
a &= -\frac{e}{m_e} \mathbf{E} \\
\mathbf{v} &= \mathbf{v}_0 + \mathbf{a}t \\
x &= x_0 + \mathbf{v}_0t + \frac{1}{2}a t^2
\end{align*}
\]
An electron accelerator

- A simple accelerator makes use of two uniformly charged plates of metal with opposite charge.

- An electron released from the negatively charged surface is repelled by negative charge and attracted by positive charge so accelerates.
An electron gun with a hot filament electron source

- Heating a wire can cause electrons to escape the metal of the wire.
- Heating is done by pushing electrons through the wire as an electrical current. Their microscopic collisions within the metal provide the thermal energy.
Cathode ray tube

- A CRT uses a hot wire electron source and simple accelerator to form an electron beam.

- Electromagnets and electrostatic deflection plates serve to point the beam at a phosphor coated screen creating light.

- The beam may be turned on and off and its position on the screen changed rapidly.