Lecture 25

Jupiter and 4 of its moons under the influence of gravity

Goal: To use Newton’s theory of gravity to analyze the motion of satellites and planets.

Newton’s Universal “Law” of Gravity

Newton proposed that every object in the universe attracts every other object.

The Law: Any pair of objects in the Universe attract each other with a force that is proportional to the products of their masses and inversely proportional to the square of their distance.

\[ F = \frac{G m_1 m_2}{r^2} \]

“Big” G is the Universal Gravitational Constant

\[ G = 6.673 \times 10^{-11} \text{ Nm}^2/\text{kg}^2 \]

Two 1 Kg particles, 1 meter away

\[ F = 6.67 \times 10^{-11} \text{ N} \]

(About 39 orders of magnitude weaker than the electromagnetic force.)

The force points along the line connecting the two objects.

Little g and Big G

Suppose an object of mass \( m \) is on the surface of a planet of mass \( M \) and radius \( R \). The local gravitational force may be written as

\[ F_G = mg \text{ } \text{surface} \]

where we have used a local constant acceleration:

\[ \vec{F}_{\text{on } m} \text{ by } M = -\frac{mGM}{R^2}\hat{r}_{M \text{ on } m} \]

\[ g \text{ } \text{surface} = \frac{GM}{R^2} \]

On Earth, near sea level, it can be shown that \( g_{\text{surface}} = 9.8 \text{ m/s}^2 \).

Exam results

- Mean/Median: 59/100
- Nominal exam curve
  - A: 81-100
  - AB: 71-80
  - B: 61-70
  - BC: 51-60
  - C: 31-50
  - D: 25-30
  - F: below 25

Table 13.1

<table>
<thead>
<tr>
<th>Altitude ( h ) (km)</th>
<th>( g ) (m/s²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 000</td>
<td>7.33</td>
</tr>
<tr>
<td>2 000</td>
<td>5.68</td>
</tr>
<tr>
<td>3 000</td>
<td>4.55</td>
</tr>
<tr>
<td>4 000</td>
<td>3.70</td>
</tr>
<tr>
<td>5 000</td>
<td>3.08</td>
</tr>
<tr>
<td>6 000</td>
<td>2.60</td>
</tr>
<tr>
<td>7 000</td>
<td>2.23</td>
</tr>
<tr>
<td>8 000</td>
<td>1.93</td>
</tr>
<tr>
<td>9 000</td>
<td>1.69</td>
</tr>
<tr>
<td>10 000</td>
<td>1.49</td>
</tr>
<tr>
<td>50 000</td>
<td>0.15</td>
</tr>
<tr>
<td>( \infty )</td>
<td>0</td>
</tr>
</tbody>
</table>
At what distance are the Earth’s and moon’s gravitational forces equal

- Earth to moon distance = \(3.8 \times 10^8\) m
- Earth’s mass = \(6.0 \times 10^{24}\) kg
- Moon’s mass = \(7.4 \times 10^{22}\) kg

\[
\frac{GM_E}{r_E^2} = \frac{GM_M}{r_M^2} = \frac{M_M}{M_E} = 0.11
\]

\[
r_M + r_E = 3.8 \times 10^8\ m
\]

\[
r_E = 3.8 \times 10^8 \text{ m/1.11} = 3.4 \times 10^8\ m
\]

The gravitational “field”

To quantify this invisible force (action at a distance) even when there is no second mass we introduce a construct called a gravitational “force” field.

The presence of a Gravitational Field is indicated by a field vector representation with both a magnitude and a direction.

\[
\vec{g} = -\frac{GM}{r^2} \hat{r}
\]

Gravitational Potential Energy

Recall for 1D: 

\[
U(x) = -F(x) \cdot dx
\]

\[
W = \int F_G \cdot dr \approx \frac{Gm_1m_2}{r_f}
\]

When two isolated masses \(m_1\) and \(m_2\) interact over large distances, they have a gravitational potential energy of

\[
U(r) = -\frac{Gm_1m_2}{r}
\]

The “zero” of potential energy occurs at \(r = \infty\), where the force goes to zero.

Note that this equation gives the potential energy of masses \(m_1\) and \(m_2\) when their centers are separated by a distance \(r\).

A plot of the gravitational potential energy

\[
U(r) = -\frac{Gm_1m_2}{r}
\]

- \(U = 0\) at \(r = \infty\)
- We use infinity as reference point for \(U(\infty) = 0\)
- This very different than
- \(U(r) = mg(r_f)\)
- Referencing infinity has some clear advantages
- All \(r\) allow for stable orbits but if the total mechanical energy, \(K+U > 0\), then the object will “escape”.

It is the same physics

Shifting the reference point to \(R_E\):

\[
U(R_E + h) = U(R_E) - \frac{Gm_1m_2}{R_E} + \frac{Gm_1m_2}{R_E + h}
\]

\[
= \frac{Gm_1m_2}{R_E} \left[ \frac{h}{R_E} + \frac{R_E + h}{R_E} \right] = \frac{Gm_1m_2}{R_E} \left[ \frac{R_E}{h} + h \right] = \frac{Gm_1m_2}{R_E} \left[ \frac{h}{R_E} + \frac{R_E}{h} \right]
\]
Gravitational Potential Energy with many masses

1. If there are multiple particles

\[ \mathcal{U} = \mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3 = -G \left( \frac{m_1 m_2}{r_1^2} + \frac{m_1 m_3}{r_1^2} + \frac{m_2 m_3}{r_2^2} \right) \]

1. Neglects the potential energy of the many mass elements that make up \( m_1, m_2 \) and \( m_3 \) (aka the “self energy”)

Dynamics of satellites in circular orbits

1. Circular orbit of mass \( m \) and radius \( r \)

\[ F = G \frac{M m}{r^2} \]

\[ \frac{1}{2} m v^2 = \frac{G M}{r} \]

\[ U = -\frac{G M m}{r} \]

\[ K = \frac{1}{2} m v^2 = \frac{1}{2} G M m / r \]

This is a general result

Total Mechanical Energy:

\[ E = K + U = \frac{1}{2} G M m / r - G M m / r \]

Changing orbits

1. With man-made objects we need to change the orbit

Examples:

1. Low Earth orbit to geosynchronous orbit
2. Achieve escape velocity

1. We must increase the potential energy but one can decrease the kinetic energy consistent with

\[ U_{\text{G}}(r) = -\frac{G m}{r} \]

\[ K(r) = -\frac{1}{2} U_{\text{G}}(r) = \frac{G m}{2 r} \]

\[ E(r) = U_{\text{G}}(r) + K(r) = -\frac{G M m}{2 r} \]

Changing orbits

1. Low Earth orbit to geosynchronous orbit

\[ E(r) = U_{\text{G}}(r) + K(r) = -\frac{G M m}{2 r} \]

A 470 kg satellite, initially at a low Earth orbit of \( h_i = 280 \text{ km} \) is to be boosted to geosynchronous orbit \( h_f = 35800 \text{ km} \)

What is the minimum energy required to do so?

\[ E_f = \frac{1}{2} \times 10^{10} \text{ J} \]

Escaping Earth orbit

1. Exercise: suppose an object of mass \( m \) is projected vertically upwards from the Earth’s surface at an initial speed \( v \), how high can it go? (Ignore Earth’s rotation)

\[ E_i = U_{\text{G}}(R_E) + \frac{1}{2} m v_i^2 \]

\[ E_f = U_{\text{G}}(R_E + h) + 0 \]

\[ E_f = E_i = U_{\text{G}}(R_E + h) + 0 \]

\[ -G M m (R_E + h) = -G M m R_E + \frac{1}{2} m v_i^2 \]

\[ -1 / (R_E + h) + 1 / R_E = \frac{1}{2} m v_i^2 / (G m M) \]

\[ h = (R_E^2 + h R_E) v_i^2 / (2 G M) \]

\[ h(1 - R_E v_i^2 / (2 G M)) = R_E^2 v_i^2 / (2 G M) \]

\[ h = \frac{R_E^2 v_i^2}{(2 G M - R_E v_i^2)} \]
Escaping Earth orbit

Exercise: suppose an object of mass $m$ is projected vertically upwards from the Earth’s surface at an initial speed $v$, how high can it go? (Ignore Earth’s rotation)

$$h = \frac{R_E^2 v_i^2}{(2GM - R_E v_i^2)}$$

$$2GM - R_E v_i^2 = 0 \quad \text{implies infinite height}$$

$$v_{\text{Escape}} = \sqrt{\frac{2GM}{R_E}} = 11.2 \text{ km/s}$$

Some interesting numbers

1. First Astronautical Speed
   - Low Earth orbit: $v = 7.9 \text{ km/s}$
2. Second Astronautical Speed
   - Escaping the Earth: $v = 11 \text{ km/s}$
3. Third Astronautical Speed
   - Escaping Solar system: $v = 42 \text{ km/s}$

Next time

1. Chapter 14 sections 1 to 3, Fluids