43) Starting from the definitions $\hat{J}_\pm = \hat{J}_x \pm i \hat{J}_y$ and basic angular momentum commutation relations, $[\hat{J}_z, \hat{J}_\pm] = i \hbar \hat{J}_\pm$, find an expression for the product $\hat{J}_+ \hat{J}_-$ in terms of the operators $\hat{J}_2$ and $\hat{J}_z$. Your result may be of use in the next problem.

44) We know that $\hat{J}_- |j, m_j \rangle = C |j, m_j - 1 \rangle$. Work out the normalization constant $C$ under the assumption that all the $|j, m_j \rangle$ kets are normalized. HINTS: Take the inner product of $\hat{J}_- |j, m_j \rangle$ with itself. Inside the bra, write out $\hat{J}_-$ in terms of $\hat{J}_x$ and $\hat{J}_y$, and then use the fact that $\hat{J}_x$ and $\hat{J}_y$ are Hermitian to move the operators over to the ket side. Then use your result from problem 43 to evaluate the bracket. The final correct answer is given in Eq. (5.55).

45) This is a problem to derive and/or verify formulas for the components of $\vec{L}$ in spherical coordinates. The operators are already known in rectangular coordinates, $L_x = \hbar (y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y})$, etc. To convert from rectangular to spherical coordinates use the chain rule; for example, $\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial \psi}{\partial \theta} \frac{\partial \theta}{\partial x} + \frac{\partial \psi}{\partial \phi} \frac{\partial \phi}{\partial x}$.

(a) Work out the formula for $L_z$ starting from the expressions $r = \sqrt{x^2 + y^2 + z^2}$, $\phi = \tan^{-1}(y/x)$ and $\theta = \tan^{-1}(z/\sqrt{x^2 + y^2})$. The final correct answer is

$$L_z = \hbar \frac{\partial}{\partial \phi}.$$ 

(b) The correct expressions for $L_x$ and $L_y$ in spherical coordinates are as follows:

$$L_x = \hbar \left[ -\sin \phi \frac{\partial}{\partial \theta} \sin \theta - \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right],$$

$$L_y = \hbar \left[ \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right].$$

Starting from $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, and $z = r \cos \theta$, convert the $L_x$ formula back to rectangular coordinates and show that you get the expected result. You do not need to do anything for $L_y$.

46) Starting from the results of problem 45 show that the operator $L^2 = L_x^2 + L_y^2 + L_z^2$ is just

$$L^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right].$$

47) In Section 5.4, Zettili works out matrices for $\hat{J}_x$, $\hat{J}_y$ and $\hat{J}_z$ for $j = 1$. Do the same for $j = \frac{3}{2}$. Its easiest to first find $\hat{J}_+$ and $\hat{J}_-$, and then construct $\hat{J}_x$ and $\hat{J}_y$ from those results.