20) A proton \((mc^2 = 938 \text{ MeV})\) with kinetic energy \(E = 1 \text{ MeV}\) encounters a potential barrier

\[
V(x) = \begin{cases} 
0 & \text{for } x < R \\
\frac{\ell(\ell+1)\hbar^2}{2m\varepsilon^2} & \text{for } x > R 
\end{cases}
\]

Use Eq. (4.71) to estimate the transmission probability through this barrier for \(\ell = 5\) and \(R = 4 \text{ fm}\). This potential represents the “angular momentum barrier” that a proton encounters as it interacts with a nucleus.

21) Consider a conventional square well potential

\[
V(x) = \begin{cases} 
0 & \text{for } -\frac{a}{2} < x < \frac{a}{2} \\
V_0 & \text{elsewhere}
\end{cases}
\]

with \(V_0 = 1 \text{ eV}\) and \(a = 1.0 \text{ nm}\). The particle bound in the well is an electron. Find the energies of the two lowest states.

22) A particle of mass \(m\) is confined in a potential

\[
V(x) = \begin{cases} 
\infty & \text{for } x < 0 \\
0 & \text{for } 0 < x < a \\
V_0 & \text{for } x > a.
\end{cases}
\]

(a) Write down the form of the wave function in each region and match the solutions at \(x = a\). We are assuming here that \(E < V_0\).

(b) How many bound states are there if \(V_0\) has the value \(32\hbar^2/ma^2\).

23) We use the “commutator” symbol \([\hat{A}, \hat{B}]\) to stand for \(\hat{A}\hat{B} - \hat{B}\hat{A}\). Prove the following commutator identities. Remember that you can not interchange the order of operators.

(a) \([\hat{A} + \hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]\).

(b) \([\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}\).

(c) \([\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}\).

24) In class we defined operators

\[
\hat{a} = \frac{1}{\sqrt{2}} \left[ \left( \frac{\sqrt{km}}{\hbar} \right)^{\frac{1}{2}} \hat{x} + i \left( \frac{1}{\hbar \sqrt{km}} \right)^{\frac{1}{2}} \hat{p} \right] \quad \text{and} \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}} \left[ \left( \frac{\sqrt{km}}{\hbar} \right)^{\frac{1}{2}} \hat{x} - i \left( \frac{1}{\hbar \sqrt{km}} \right)^{\frac{1}{2}} \hat{p} \right]
\]

(a) Show that \([\hat{a}, \hat{a}^\dagger] = 1\).

(b) In class we found that \(H = [\hat{a}^\dagger \hat{a} + \frac{1}{2}]\hbar\omega\). Use this together with the equation from part (a) and the identities from Problem 23 to find \([H, \hat{a}^\dagger]\).

(c) Suppose \(H|\psi\rangle = E|\psi\rangle\). Use your result from part (b) to show that \(\hat{a}^\dagger|\psi\rangle\) is then also an eigenstate of \(H\). What is the eigenvalue?
25) In this problem $\psi_0(x)$, $\psi_1(x)$ and $\psi_2(x)$ are the first three energy eigenfunctions for the harmonic oscillator.

(a) Using equations from class or the text, write out each wave functions in full detail including the normalization constant.

(b) Show by explicit calculation that $\psi_0(x)$ and $\psi_2(x)$ are orthogonal.

26) A particle of mass $m$ is confined in a harmonic oscillator well. Suppose that at time $t=0$ the particle is in a quantum state described by the “wave packet”

$$\Psi(x,t=0) = \sqrt{\frac{2}{3}} \left[ \frac{a}{\sqrt{\pi}} \right]^\frac{4}{3} \left[ 1 + ax \right] e^{-a^2 x^2 / 2}$$

where $a^2 = \left( \sqrt{k}m/\hbar \right)$.

(a) Find the full time dependent wave function (i.e. $\Psi$ for all $t$) by expanding $\Psi$ in terms of energy eigenfunctions $\psi_0$ and $\psi_1$.

(b) Find $\langle x \rangle$ as a function of time for this wave function.