57) The radial wave functions for states in hydrogen are generally not too complicated. In particular, the lowest-energy state for each \( l \) (i.e. the state with \( n = l+1 \)) has a radial wave function
\[
R(r) = N_l r^l e^{-r/a_0},
\]
where \( a_0 \) is the Bohr radius. Find the expectation values of \( 1/r \) and of \( 1/r^2 \) for these states. We will use the results in the next problem.

58) Calculate the relativistic correction to the kinetic energy,
\[
\Delta E_{rel} = \langle \psi_{nlm} | -\frac{p^4}{8 m^4 c^2} | \psi_{nlm} \rangle
\]
for the states considered in Problem 57 above. Your result should be consistent with the general formula given in Eq. (9.89). [Hints: The states \( \psi_{nlm} \) are eigenfunctions of our original nonrelativistic Hamiltonian \( H_0 \) so you can replace \( \frac{p^2}{2m} \psi_{nlm} \) by \( (H_0 - V)\psi_{nlm} = (E_n^0 - V)\psi_{nlm} \). Similarly, as we discussed in class, \( (\frac{p^2}{2m})^2 \psi_{nlm} \) can be replaced by \( (E_n^0 - V)^2 \psi_{nlm} \).]

59) In this problem we will construct some of the \( J^2 \), \( J_z \) eigenfunctions for the \( 2p \) state in hydrogen. The basis functions for the problem are the functions \( Y_{l}^{m}(m_s) \), and according to what we said in class, the state \( |j, m \rangle = |\frac{3}{2}, \frac{3}{2} \rangle \) is just \( Y_{1}^{1}(\frac{3}{2}) \).

(a) Find the state \( |\frac{1}{2}, -\frac{3}{2} \rangle \) by applying the lowering operator \( J^- \) to \( |\frac{3}{2}, \frac{3}{2} \rangle \). To find the state, write \( J^- = L^- + S^- \) and then use Equation (5.55) of the text, to decide what \( L^- \) and \( S^- \) do to \( Y_{1}^{1}(\frac{3}{2}) \).

(b) Normalize the state you found in part (a) and then find \( |\frac{1}{2}, \frac{1}{2} \rangle \) by using the fact that \( |\frac{1}{2}, -\frac{1}{2} \rangle \) and \( |\frac{1}{2}, \frac{1}{2} \rangle \) must be orthogonal.

60) A particle of mass \( m \) in an infinite square well (that extends from 0 to \( L \)) is subject to a perturbation \( H'(x) = A \cos \frac{2 \pi}{L} x \).

(a) Find the first-order shift of the ground state energy.

(b) Find a general formula for the first-order energy shift of the excited states, \( n \geq 2 \). Remember that the square well energy eigenstates are given by \( \phi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n \pi}{L} x \).

(c) In first order, the wave functions have the form \( \psi_n = \phi_n + \sum_k a_k \phi_k \). Determine which of the \( a_k \) coefficients will be non-zero for arbitrary \( n \). [Hint: There is a trig identity for \( \cos x \sin y \) that will let you write \( H' \phi_n \) as an expansion over the energy eigenstates.]

61) The energy eigenfunctions for the 2-dimensional harmonic oscillator, \( V = \frac{1}{2} k (x^2 + y^2) \), are of the form \( \phi_{nm} = \phi_n(x) \phi_m(y) \) where \( \phi_n(x) \) and \( \phi_m(y) \) are ordinary one-dimensional harmonic oscillator wave functions
\[
\phi_0(x) = \left[ \frac{a}{\sqrt{\pi}} \right]^\frac{1}{2} e^{-a^2 x^2/2}, \quad \phi_1(x) = \left[ \frac{2 a}{\sqrt{\pi}} \right]^\frac{1}{2} a x e^{-a^2 x^2/2}, \quad \text{etc.}
\]
The energy of state \( \phi_{nm} \) is \( (n + m + 1) \hbar \omega \). Find the first-order energy shifts and the zero-order energy eigenfunctions for the states \( \phi_a = \phi_0(x) \phi_1(y) \) and \( \phi_b = \phi_1(x) \phi_0(y) \) (which are initially degenerate) resulting from a perturbation of the form \( H' = bxy \).