Physics 207 Exam 2 (Hokin)
April 13, 2011

Name ________________
TA __________________

Problem 1 (carbon-14 decay) __________
Problem 2 (rolling object race) __________
Problem 3 (binary stars) __________
Problem 4 (mass and spring) __________
Problem 5 (water stream) __________
TOTAL __________

Constants, etc.

\[ G = 6.67 \times 10^{-11} \text{Nm}^2/\text{kg}^2 \] (gravitational constant)
\[ c = 3.0 \times 10^8 \text{m/s} \] (speed of light)
\[ g = 9.8 \text{m/s}^2 \] (near-Earth gravitational acceleration)
\[ m_e = 5.48 \times 10^{-4} u = 9.11 \times 10^{-31} \text{kg} \] (mass of electron)
\[ m_e c^2 = 511 \text{keV} \] (mass of electron in energy units)
\[ M_{\text{sun}} = 2.0 \times 10^{30} \text{kg} \] (mass of Sun)
\[ M_{\text{moon}} = 7.4 \times 10^{22} \text{kg} \] (mass of Moon)
\[ M_{\text{earth}} = 6.0 \times 10^{24} \text{kg} \] (mass of Earth)
\[ D_{\text{sun}} = 1.5 \times 10^{11} \text{m} \] (Earth-Sun distance)
\[ D_{\text{moon}} = 3.8 \times 10^8 \text{m} \] (Earth-Moon distance)
\[ R_{\text{earth}} = 6.4 \times 10^6 \text{m} \] (Earth radius)
\[ \rho_{\text{air}} = 1.2 \text{kg/m}^3 \] (mass density of air)
\[ \rho_w = 1000 \text{kg/m}^3 \] (mass density of water)

\[ ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \] (quadratic equation)
1. The Carbon-14 ($^{14}$C) nucleus beta decays to stable Nitrogen-14 ($^{14}$N) by emitting an electron and a neutrino. The neutrino has zero mass, but it does carry momentum and energy (special relativity). Its kinetic energy and momentum are simply related by $K_n = p_n c$, where $c$ is the speed of light. The total energy released in the $^{14}$C decay is $E = 156.5 \text{ keV} = 2.50 \times 10^{-14} \text{ J}$.

In a particular decay, the electron and neutrino are emitted in exactly opposite directions. Assume that the $^{14}$N recoil momentum and kinetic energy may be ignored, i.e. the $^{14}$N nucleus remains stationary.

a) (5 pts) Write an equation for conservation of energy in terms of $E$, the electron kinetic energy $K_e$, and the neutrino kinetic energy $K_n$.

$$E = K_e + K_n$$

b) (5 pts) Write an equation for conservation of momentum in terms of $K_e$, $K_n$, and $m_e$.

Momentum equals opposite

$$p_n = -p_e$$

$$\frac{K_n}{c} = \sqrt{2m_e K_e} \quad \text{or} \quad K_n = \frac{1}{2} m_e c^2 K_e$$

$$p_e = \frac{K_e}{2m_e}$$

$$v_e = \frac{K_e}{m_e}$$

C) (10 pts) Solve for $K_e$ and $K_n$.

Up to

$$K_n = \frac{2m_e c^2 K_e}{2m_e c^2}$$

$$E = K_e + K_n \Rightarrow 2m_e c^2 E = 2m_e c^2 K_e + 2m_e c^2 K_n$$

$$2m_e c^2 E = K_n + 2m_e c^2 K_n$$

$$K_n^2 + 2m_e c^2 K_n - 2m_e c^2 E = 0$$

Quadratic formula

$$K_n = -\frac{1}{2} m_e c^2 \pm \sqrt{\frac{1}{4} m_e^2 c^4 + \frac{1}{2} m_e c^2 E}$$

$$\frac{K_n}{m_e c^2} = -1 + \frac{1}{2} \frac{1 + 2E/m_e c^2}{1 + 2E/m_e c^2} = 0.270$$

$$K_n = 137.9 \text{ keV} \quad K_e = 18.6 \text{ keV}$$

$$= 2.2 \times 10^{-14} \text{ J} \quad = 3.0 \times 10^{-14} \text{ J}$$
2. A solid cylinder and a solid sphere, both of radius $R$ and mass $M$, are released from rest on a ramp of angle $\theta$ and height $h$.

a) (10 pts) Find the speeds $v_{cyl}$ and $v_{sph}$ when they reach the bottom of the ramp in terms of the given quantities and $g$.

Sphere: $Mgh = \frac{1}{2}MV_{sph}^2 + \frac{1}{2}I_{sph}\omega^2$

$Mgh = \frac{1}{2}MV_{sph}^2 + \frac{1}{2}MR^2\frac{\omega^2}{5}$

$gh = \frac{1}{2}V_{sph}^2\left(1 + \frac{2}{5}\right) = \frac{g^2h^2}{10}$

$\omega = \frac{V_{sph}}{R}$

$\Rightarrow V_{sph} = \sqrt{\frac{10}{7}gh}$

Cylinder: $I = \frac{1}{2}mr^2 \Rightarrow gh = \frac{1}{2}V_{cyl}^2\left(1 + \frac{1}{2}\right) = \frac{g^2h^2}{4}$

$\Rightarrow V_{cyl} = \sqrt{\frac{4}{3}gh}$

b) (5 pts) Find the times $t_{cyl}$ and $t_{sph}$ that it takes for the cylinder and sphere to reach the bottom of the ramp.

Ramp length $s = \frac{h}{\sin \theta}$ at avg speed $v' = \frac{\omega}{\sin \theta}$

$t_{sph} = \frac{2h}{\sin \theta} \Rightarrow \sqrt{\frac{7}{10gh}} = \frac{1}{\sin \theta} \sqrt{\frac{14}{5gh}}$

$t_{cyl} = \frac{2h}{\sin \theta} \sqrt{\frac{3}{16gh}} = \frac{1}{\sin \theta} \sqrt{\frac{3}{5gh}}$

$c)$ (5 pts) Which object reaches the bottom first? Why? Name a rolling object of the same mass and radius that would be slower than both the cylinder and sphere.

The sphere reaches first because it has smaller $I$, so more energy goes into translation.

A hollow ring would be even slower.
3. A binary star system, consisting of two stars each with the mass of our sun, has orbital period of 90 days as the stars orbit the system's center of mass at radius $R$.

\[ F = \frac{GM M}{(2R)^2} = \frac{GM^2}{4R^2} = \frac{GM^2}{4R^2} = M \omega^2 R \quad \omega = \frac{2\pi}{T} \]

\[ R^3 = \frac{GM}{4\omega^2} = \frac{GM}{4 \left( \frac{2\pi}{T} \right)^2} = \frac{GM}{16 \pi^2 T^2} \]

\[ R = \left( \frac{GM}{16 \pi^2 T^2} \right)^{\frac{1}{3}} = 3,704 \times 10^{10} \text{ m} \]

b) (5 pts) Determine the total gravitational potential energy $U_g$ of the system.

\[ U_g = -\frac{GM^2}{2R} = -3.562 \times 10^{39} \text{ J} \]

c) (5 pts) Determine the total kinetic energy $K$ of the system.

\[ K = \frac{1}{2} U_g = 1.781 \times 10^{39} \text{ J} \]
4. A spring with spring constant $k$ is suspended vertically. A mass $m$ is attached and released from the unstretched position of the spring. The mass then oscillates up and down with the lowest point 20 cm below the release position.

\begin{align*}
\text{a) (5 pts)} & \quad \text{Write an expression for the total energy of the system at the top of the oscillation.} \\
& \quad V = 0 \quad \text{at top} \implies K = 0 \\
& \quad V = \frac{1}{2} k y^2 = 0 \quad \text{at top (spring relaxed)} \\
& \quad y = 0 \implies V = 0 \\
& \quad E = 0 \quad (\text{using my choice of } y) \\
\text{b) (5 pts)} & \quad \text{Write an expression for the total energy of the system at the bottom of the oscillation.} \\
& \quad U = 0 \quad \text{at bottom} \implies K = 0 \\
& \quad U_p = \frac{1}{2} k y_{\text{bot}}^2 \\
& \quad U_g = -mg y_{\text{bot}} \\
& \quad E = \frac{1}{2} k y_{\text{bot}}^2 - mg y_{\text{bot}} = 0 \\
\text{c) (10 pts)} & \quad \text{Solve for the oscillation frequency in Hz.}
\end{align*}

\begin{align*}
& \quad \frac{1}{2} k y_{\text{bot}}^2 = mg y_{\text{bot}} \\
& \quad \implies w^2 = \frac{k}{m} = \frac{2g}{y_{\text{bot}}} \\
& \quad w = \sqrt{\frac{2g}{y_{\text{bot}}}} = 9.9 \text{ rad/sek} \\
& \quad f = \frac{w}{2\pi} = 1.6 \text{ Hz}
\end{align*}
5. A large drum is filled with water to height \( h \), and filled continually to maintain the water level. A small hole is punched into the side of the drum at height \( y \).

\[ \begin{align*}
\frac{1}{2} m v^2 + \frac{1}{2} k^2 y^2 + c &= \frac{1}{2} m v_0^2 + \frac{1}{2} k y^2 + \frac{1}{2} k x^2 \\
\frac{1}{2} x^2 &= 2g (h-y) \\
v &= \sqrt{2g (h-y)}
\end{align*} \]

a) (5 pts) Find an expression for the speed of the water \( v \) leaving the hole in terms of the given quantities and \( g \).

b) (5 pts) Find an expression for the distance \( x \) away from the drum where the water stream hits the ground in terms of \( v \) and the given quantities. (Use physics from early in the course.)

\[ x = vt \Rightarrow t = x/v \]

\[ y = \frac{1}{2} g t^2 = \frac{1}{2} g (\frac{x}{v})^2 = \frac{1}{2} g \frac{x^2}{v^2} \Rightarrow x^2 = \frac{2vy^2}{g} \]

\[ x = \sqrt{\frac{2vy^2}{g}} \]

c) (10 pts) Find the hole height \( y_{\text{max}} \) that results in the maximum range of the water stream, \( x_{\text{max}} \), and find \( x_{\text{max}} \) as well.

Maximize \( x(y) \) : \( \frac{1}{2} x^2 = 2g (h-y) \)

\[ \Rightarrow x^2 = \frac{2y}{g} (2g (h-y)) = 4y (h-y) \]

Minimize \( y (h-y) = hy - y^2 \)

\[ \frac{dy}{dy} = 0 = h - 2y \Rightarrow y_{\text{max}} = \frac{h}{2} \]

\[ x_{\text{max}}^2 = 4 \frac{h}{2} (h-\frac{h}{2}) = 4 \frac{h^2}{4} = h \]

\[ x_{\text{max}} = h \]