Phys 201, Lecture 5

Chapter 3:
Motion in Two and Three Dimensions

3-D Kinematics

- The position, velocity, and acceleration of a particle in 3 dimensions can be expressed as:

\[ \mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \]
\[ \mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k} \quad \text{(i, j, k unit vectors)} \]
\[ \mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \quad \text{(time-independent)} \]
- We have already seen the 1-D kinematics equations:

| \[ x = x(t) \] | \[ y = \frac{dx}{dt} \] | \[ a = \frac{d^2x}{dt^2} \] |

3-D Kinematics

- For 3-D, we may simply apply the 1-D equations to each of the component equations.

\[ x = x(t) \quad y = y(t) \quad z = z(t) \]
\[ v_x = \frac{dx}{dt} \quad v_y = \frac{dy}{dt} \quad v_z = \frac{dz}{dt} \]
\[ a_x = \frac{d^2x}{dt^2} \quad a_y = \frac{d^2y}{dt^2} \quad a_z = \frac{d^2z}{dt^2} \]

These can be combined into the vector equations:

\[ \mathbf{a} = \mathbf{a}(t) \]
\[ \mathbf{v} = \frac{d\mathbf{r}}{dt} \]
\[ \mathbf{r} = \mathbf{r}(t) \]

Displacement of a particle in two dimensions

\[ r = \sqrt{x^2 + y^2} \]
\[ y = r \sin \theta \]
\[ x = r \cos \theta \]
Change of displacement of a moving particle

Average velocity over time interval $\Delta t$:

\[
\bar{v} = \frac{\Delta r}{\Delta t}
\]

Instantaneous velocity:

\[
v(t) = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \frac{dr}{dt}
\]

in the tangential direction on the $r$-$t$ graph, independent of the $r$ direction.

Magnitude of velocity vector:

\[
v = \sqrt{v_x^2 + v_y^2}
\]

Direction of velocity vector described by $\theta$:

\[
\theta = \tan^{-1} \frac{v_y}{v_x}
\]

Relative velocity

- Velocity is defined relative to a frame of reference.

Mathematically, a vector sum:

\[
v_{\text{resultant}} = v_{\text{relative}} + v_{\text{frame}}
\]
Example 3-2: Flying plane in wind

Wind blows east (along x) with velocity $v_{AG} = 90$ km/h.

Pilot of plane that flies 200 km/h wishes to fly due north.

What direction should he point?

To go due north ($y$), this east ($x$)-component of the resultant velocity must be zero:

$$\sin \theta = \frac{v_{pA}}{v_{pG}} = \frac{90 \text{ km/hr}}{200 \text{ km/hr}}$$

$$\theta = 0.47 \text{ radians} \approx 27^\circ \text{ W of N}$$

The velocity magnitude:

$$v_{pG} = v_{pA} \cos \theta = 180 \text{ km/hr}$$

Question

Three swimmers can swim equally fast relative to the water. They have a race to see who can swim across a river in the least time. Relative to the water, Beth (B) swims perpendicular to the flow, Ann (A) swims upstream, and Carly (C) swims downstream. Which swimmer wins the race?

A) Ann  B) Beth  C) Carly

Time to get across = width of river/perpendicular component of velocity.

Beth has the largest perpendicular component of velocity.

correct
Question (seagull)

A seagull flies through the air with a speed of 10 m/s in the absence of wind. Assume it can only make the same effort while flying in wind. It makes a daily round-trip to an island one km from shore. Compare the time it takes for the seagull to fly on a calm day to the time it takes when the wind is blowing constantly towards the shore at 5 m/s.

a. The round-trip time is the same with and without the wind
b. The round-trip time is always longer with the wind
c. The round-trip time can be shorter than without the wind

Total round trip time in the absence of wind is $2 \times \frac{1000 \text{ m}}{10 \text{ m/s}} = 200 \text{ s}$.

In the presence of wind, the seagull’s speed going towards shore is 15 m/s and away from shore is 5 m/s. The total time to go out to the island & the time to return: $(1000 \text{ m})/(5 \text{ m/s}) = 200 \text{ s}$, and $(1000 \text{ m})/(15 \text{ m/s}) = 67 \text{ s}$, so the total time in the presence of wind is 267 s.

In general: $t = \frac{2 \times d}{v - v_w^2/v^2}$

Acceleration vectors

Average acceleration over time interval $\Delta t$:

$\bar{a} = \bar{a} = \frac{\Delta \vec{v}}{\Delta t}$

Instantaneous acceleration:

$\ddot{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$

in the tangential direction on the v-t graph, Not tangential to r-t graph, independent of the r, v directions.

Simplify to 2-D Kinematics

Planar 3-D problems can be reduced to 2-D’s:

Choose y axis to be along direction of acceleration.
Choose x axis to be along the “other” direction of motion.
⇒ Perpendicular to y, but co-planar with the trajectory.

Projectile motion

For projectile motion:

Horizontal acceleration is zero
⇒ horizontal velocity is constant

Vertical acceleration is $-g$
(constant in magnitude g, directed downward)

The horizontal and vertical motions are independent, except that the object stops moving both horizontally and vertically at the instant it hits the ground (or some other object, which determines y-range).
Without air resistance, an object dropped from a plane flying at constant speed in a straight line will

A. Quickly lag behind the plane.
B. Remain vertically under the plane.
C. Move ahead of the plane.

There is no acceleration in the horizontal direction – object continues to travel with the same horizontal velocity (same as the plane). Due to gravitational acceleration, the object accelerates downward, so its speed downwards increases.

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**Vertical and horizontal motions are independent**

For projectile motion, the vertical positions of these two balls are the same at each time:  
\[ x = v_0 T \]

Vertical motion and horizontal motion are independent:  
\[ y = y_0 + v_{0y} T + \frac{1}{2}(-g)T^2 \]

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**Horizontal range of a projectile**

The horizontal range is the product of the horizontal speed (x component of the velocity) and the total time that the projectile is in the air.

If object starts at height \( y_0 = 0 \), then \( T \), the time in the air, is determined by finding when it reaches height \( y = 0 \) again:

\[ 0 = v_{0y} T - \frac{1}{2} g T^2 \]

The two solutions of this equation are \( T = 0 \) (as expected) and \( T = \frac{2v_{0y}}{g} = \frac{2v_0 \sin(\theta)}{g} \).

The horizontal range is then \( v_0 T = \left( v_0 \cos(\theta) \right) \left( \frac{2v_0 \sin(\theta)}{g} \right) = \frac{v_0^2 \sin(2\theta)}{g} \).

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The range of a projectile depends on initial angle. Starting at ground level (\( y = 0 \)), the range is maximized for \( \theta = 45^\circ \).
If a projectile lands at an elevation lower than the initial elevation, the maximum horizontal displacement is achieved when the projection angle is somewhat less than 45°.

A battleship simultaneously fires two shells at enemy ships from identical cannons. If the shells follow the parabolic trajectories shown, which ship gets hit first?

1. Ship A
2. Ship B
3. Both at the same time

The higher the shell flies, the longer the flight takes:

\[
T = \frac{2v_0\sin \theta}{g}
\]

Acceleration for a general curved path

Instead of considering

\[
a = a_t + a_c
\]

(time-independent)

Decomposed into:

\[
a = a_t + a_c
\]

the tangential acceleration: \( a_t = \frac{dv}{dt} \)

and centripetal acceleration: \( a_c \)

Velocity: \( r \) change rate

\[
\mathbf{v}(t + \Delta t) = \mathbf{v}(t) + \frac{\Delta \mathbf{r}}{\Delta t} = \mathbf{v}(t) + \mathbf{a} \Delta t
\]

Acceleration: \( v \) change rate

Centripetal acceleration:

\[
\frac{v^2}{r} \Delta t
\]

Thus,

\[
a_c = \frac{v^2}{r}
\]

Directing toward center!
Example 3-4  Acceleration for uniform circular motion.

Initial velocity has magnitude $v$ and points due east.

Final velocity has same magnitude $v$ and points due north.

Velocity has changed → particle is accelerating!

Acceleration constant in magnitude, direction changing with time.

\[
\text{average acceleration} = \frac{\Delta \mathbf{v}}{\Delta t}
\]

the tangential acceleration: $a_t = \frac{dv}{dt} = 0$

the centripetal acceleration: $a_c = \frac{v^2}{r}$

Non-uniform circular motion:

A pendulum

*In general, $a = a_t + a_c$

*Both non-zero.*