Hooke's Law

Spring

Simple Harmonic Motion

Energy

Springs

- Hooke's Law: The force exerted by a spring is proportional to the distance the spring is stretched or compressed from its relaxed position.

\[ F_x = -kx \]

Where \( x \) is the displacement from the relaxed position and \( k \) is the constant of proportionality (often called "spring constant")

Simple Harmonic Motion

- We know that if we stretch a spring with a mass on the end and let it go, the mass will oscillate back and forth (if there is no friction).

- This oscillation is called Simple Harmonic Motion, and is actually easy to understand...

Simple Harmonic Motion (SHM)

\[ m \frac{d^2x}{dt^2} = -kx \]

SHM Dynamics

- At any given instant we know that \( F = ma \) must be true.

- But in this case \( F = -kx \) and \( ma = m \frac{d^2x}{dt^2} \)

So: \( -kx = ma \)

\[ \frac{d^2x}{dt^2} = -\frac{k}{m} x \]
Try the solution \( x = A \cos(\omega t) \)

This works, so it must be a solution!

\[
\frac{d^2 x}{dt^2} = -\omega^2 x
\]

The angular frequency: \( \omega \)

The period: \( T = \frac{2\pi}{\omega} \)

We just showed that \( \frac{d^2 x}{dt^2} = -\omega^2 x \) (from \( F = ma \))

has the solution \( x = A \cos(\omega t) \) but \( x = A \sin(\omega t) \) is also a solution.

The most general solution is a linear combination (sum) of these two solutions!

\[
x = B \sin(\omega t) + C \cos(\omega t) = A \cos(\omega t + \phi)
\]

\[
\frac{dx}{dt} = \omega B \cos(\omega t) - \omega C \sin(\omega t)
\]

\[
\frac{d^2 x}{dt^2} = -\omega^2 B \sin(\omega t) - \omega^2 C \cos(\omega t) = -\omega^2 x \quad \text{ok}
\]

But wait a minute...what does angular frequency \( \omega \) have to do with moving back & forth in a straight line??

The spring's motion with amplitude \( A \) is identical to the \( x \)-component of a particle in uniform circular motion with radius \( A \).

\[
x = A \cos(\omega t + \phi) \quad \text{The period: } T = \frac{2\pi}{\omega}
\]

What is the angular frequency of oscillation \( \omega \)?

What is the spring constant \( k \)?

\[
\omega = \sqrt{\frac{k}{m}}
\]

\[
\Rightarrow k = \frac{m\omega^2}{\sqrt{m}} = 800 \text{ N/m}
\]

Example

A mass \( m = 2 \text{ kg} \) on a spring oscillates with amplitude \( A = 10 \text{ cm} \). At \( t = 0 \) its speed is maximum, and is \( v = 2 \text{ m/s} \).

What is the angular frequency of oscillation \( \omega \)?

What is the spring constant \( k \)?

\[
\omega = \frac{v_{\text{max}}}{A} = \frac{2 \text{ m/s}}{0.1 \text{ m}} = 20 \text{ s}^{-1}
\]

Also:

\[
\omega = \sqrt{\frac{k}{m}}
\]

\[
\Rightarrow k = \frac{m\omega^2}{\sqrt{m}} = 800 \text{ N/m}
\]

So \( k = (2 \text{ kg}) (20 \text{ s}^{-1})^2 = 800 \text{ kg} \cdot \text{s}^2 = 800 \text{ N/m} \)
Use "initial conditions" to determine phase \( \phi \).

Suppose we are told \( x(0) = 0 \), and \( x \) is initially increasing (i.e. \( v(0) = \) positive);

\( x(0) = 0 = A \cos(\phi) \quad \implies \quad \phi = \frac{\pi}{2} \) or \(-\frac{\pi}{2}\)

\( v(0) > 0 = -\omega A \sin(\phi) \quad \implies \quad \phi < 0 \)

So \( \phi = -\frac{\pi}{2} \)

\[ x(t) = A \cos(\omega t + \phi) \]
\[ v(t) = -\omega A \sin(\omega t + \phi) \]
\[ a(t) = -\omega^2 A \cos(\omega t + \phi) \]

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**What about Vertical Springs?**

- We already know that for a vertical spring
  \[ F = -k y = -m \cdot \frac{dy}{dt}, \]
  if \( y \) is measured from the equilibrium position
  \[ \frac{d^2y}{dt^2} = -\omega^2 y \]

- So this will be just like the horizontal case:
  \[ y = A \cos(\omega t + \phi) \]
  \[ \omega = \sqrt{\frac{k}{m}} \]

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**The Simple Pendulum...**

- Recall that the torque due to gravity about the rotation (\( z \)) axis is
  \[ \tau = -mgd \quad \text{for small } \theta \]
  \[ d = \text{L} \sin \theta \approx \text{L} \theta \]
  \[ \tau = -mgL \theta \]

- But \( \tau = I \frac{d\theta}{dt} \), where \( I = mL^2 \)

  \[ \frac{d^2\theta}{dt^2} = -\omega^2 \theta \]

  \[ \omega = \sqrt{\frac{mg}{L}} \]

Differential equation for simple harmonic motion:

\[ \theta = \theta_0 \cos(\omega t + \phi) \]
**Solution:**

\[ s = A \cos(\omega t + \phi) \]

**Force:**

\[ \frac{d^2 s}{dt^2} = -\omega^2 s \]

**Energy for simple harmonic motion: \( E = K + U \)**

\[ E = \frac{1}{2} m v^2 + \frac{1}{2} k x^2 = \text{constant} \]

\[ KE_{\text{max}} = \frac{1}{2} M v^2_{\text{max}} / 2 = M a^2 A^2 / 2 = kA^2 / 2 \]

\[ PE_{\text{max}} = kA^2 / 2 \]

**Energy conservation**

\[ E_{\text{total}} = kA^2 / 2 \]