Lecture 22: 
Chapter 11 – Gravity 
Lecture 2

- Gravitational potential energy 
- Escape velocity 
- Gravitational Field of a point mass 
- Gravitational Field for mass distributions 
  - Discrete 
  - Rod 
  - Spherical shell 
  - Sphere 
- Gravitational potential energy of a system of particles 
- Black holes

Potential Energy

\[ PE = -W = \int F \cdot dr = -\int \left( -\frac{G M m}{r^2} \right) dr \]

Force 
\[ F = \frac{G M m}{r^2} \]

Work done to bring mass \( m \) from initial to final position.

Total energy 
\[ E_{\text{tot}} = K + U = \frac{1}{2} m v^2 - \frac{G M m}{r} \]
**Binding Energy**

- The absolute value of the potential energy can be thought of as the **binding energy**.
- At infinite separation, binding energy \( U = 0 \), thus unbound.
- If an external agent applies a force larger than the binding energy, the excess energy will be in the form of kinetic energy of the particles when they are at infinite separation.

**Escaping Gravity**

- Kinetic energy of the object must be greater than its gravitational potential energy.
- This defines the minimum velocity to escape.
- \( KE + PE = \text{constant} \).
- Consider case when speed is just sufficient to escape to infinity with vanishing final velocity.
- At infinity, \( KE + PE = 0 \), therefore, on Earth,
  \[
  \frac{1}{2} mv_{\text{esc}}^2 - \frac{GM_m}{R} = 0 \implies \]
  \[
  v_{\text{esc}} = \sqrt{\frac{2GM}{R}} = 11.2 \text{ km/s} = 25000 \text{ mph}
  \]

**Quiz**

- You are on the moon and you know how to calculate the escape velocity:
  \[
  v_{\text{esc}} = \sqrt{\frac{2GM}{R}}
  \]
- You find that it is 2.37 km/s.

A projectile from the moon surface will escape even if it is shot horizontally, not vertically with a speed of at least 2.37 km/s.

A) Correct
B) Not correct
Gravity near Earth’s surface...

- Near the Earth’s surface:
  - \( R_{12} = R_E = 6371 \) km
  - Won’t change much if we stay near the Earth’s surface.
    - since \( R_E \gg h, R_E + h \approx R_E \).

\[
F_g = \frac{G M_{\odot} m}{R_E^2}
\]

Variation of \( g \) with Height

<table>
<thead>
<tr>
<th>Altitude ( h ) (km)</th>
<th>( g ) (m/s(^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 000</td>
<td>7.35</td>
</tr>
<tr>
<td>2 000</td>
<td>5.98</td>
</tr>
<tr>
<td>4 000</td>
<td>4.53</td>
</tr>
<tr>
<td>5 000</td>
<td>3.98</td>
</tr>
<tr>
<td>10 000</td>
<td>2.25</td>
</tr>
<tr>
<td>8 000</td>
<td>1.95</td>
</tr>
<tr>
<td>9 000</td>
<td>1.69</td>
</tr>
<tr>
<td>10 000</td>
<td>1.49</td>
</tr>
<tr>
<td>50 000</td>
<td>0.13</td>
</tr>
<tr>
<td>( \infty )</td>
<td>0</td>
</tr>
</tbody>
</table>

This is twice the Earth radius: \( R_E = 6371 \) km
We know \( F \) should drop with \( r^2 \)
Indeed, “\( g \)” has dropped to 9.81/4 m/s\(^2\)

Gravity...

- Near the Earth’s surface...

\[
F_g = \frac{G M_{\odot} m}{R_E^2}
\]

- So \( |F_g| = mg = ma \)
  - \( a = g \)
  - All objects accelerate with acceleration \( g \) regardless of their mass!

Or: the equivalence principle: \( m \frac{d^2}{dt^2} = m \frac{d}{dt} \left( \frac{d}{dt} \right) \)
Choosing \( \Phi(R_E) = 0 \), then
\[
\Phi(h) = m g h, \quad \text{for } h \ll R_E
\]

Question

Suppose you are standing on a bathroom scale in your dorm room and it says that your weight is \( W \). What will the same scale say your weight is on the surface of the mysterious Planet X?

You are told that \( R_X \approx 20 R_E \), and \( M_X \approx 300 M_E \).

(a) 0.75 \( W \)
(b) 1.5 \( W \)
(c) 2.25 \( W \)

\[
F_g = \frac{G m M}{r^2}
\]

\[
e^{-2}\]

\[
e^{-3}\]

\[
e^{-4}\]

\[
e^{-5}\]
Gravitational Field

- Gravitational force:
  \[ \vec{F}_{12} = \frac{Gm_1m_2}{r_{12}^2} \hat{r}_{12} \]
  it is a function of space-time \((r, t)\).

- Definition of the gravitational field that will act on any masspoint:
  \[ \vec{g} = \vec{F}/m \]
  \[ \vec{g} = \sum \vec{F}_{i} \]
  Must be a function of space-time \((r, t)\) \(\rightarrow\) concept of “field”.

- If the field is caused by a mass distribution we need to sum over all masspoints as the source.

Gravitational Field

Two source mass points \(M_1, M_2\), fieldpoint in plane of symmetry

Magnitude of field due to each mass:
\[ g_i = G \frac{M_i}{r_i^2} \]

Need to add x and y component of \(g_1\) and \(g_2\)

X-component:
\[ g_x = g_{x1} + g_{x2} = 2G \frac{M}{r^2} \cos(\theta) = 2G \frac{M}{r^2} \]

Y-component is zero for symmetry reasons
\[ g_y = 0 \]
Gravitational Field

Field due to spherical symmetric mass distribution, a shell of mass $M$ and radius $R$:

Field of a spherical shell

$g = \frac{-GM}{r^2} \quad r > R$

$g = 0 \quad r < R$

Geometry: spherical shell is 0 anywhere inside (see p.384)

---

Systems with Three or More Particles

- The total gravitational potential energy of the system is the sum over all pairs of particles: simple scalar sum
- Gravitational potential energy obeys the superposition principle
- Each pair of particles contributes a term of $U_{ij}$
- The absolute value of $U_{total}$ represents the work needed to separate the particles by an infinite distance

$U_{total} = U_{12} + U_{13} + U_{23}$

$U_{total} = -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}}$

---

Potential energy of a system of masses

- What is the total potential energy of this mass system?

$U = -3G \frac{mnL}{L}$
Four identical masses, each of mass $M$, are placed at the corners of a square of side $L$. The total potential energy of the masses is equal to $-xGM^2/L$, where $x$ equals

A. 4  
B. $4 + 2\sqrt{2}$  
C. $4 + \sqrt{2}$  
D. $4 + \frac{1}{\sqrt{2}}$  
E. $2 + 2\sqrt{2}$

Black Holes

- A black hole is the remains of a star that has collapsed under its own gravitational force.
- The escape speed for a black hole is very large due to the concentration of a large mass into a sphere of very small radius.
  - If the escape speed exceeds the speed of light, radiation cannot escape and it appears black.
- The critical radius at which the escape speed equals $c$ is called the Schwarzschild radius, $R_S$.
- The imaginary surface of a sphere with this radius is called the event horizon.
  - This is the limit of how close you can approach the black hole and still escape.

Black Holes at Centers of Galaxies

- There is evidence that supermassive black holes exist at the centers of galaxies ($M=100$ million solar masses).
- Theory predicts jets of materials should be evident along the rotational axis of the black hole.

An Hubble Space Telescope image of the galaxy M87. The jet of material in the right frame is thought to be evidence of a supermassive black hole at the galaxy’s center.