Kepler's (empirical) Laws

Describe the features of planetary motions, and led to the Newton's law of gravity.

Johannes Kepler (1571–1630): a German mathematician, astronomer and astrologer; an assistant to astronomer Tycho Brahe who took a lot of data from telescopes.

Today we can understand the physical reasons for these laws …
Let’s remind us first of the geometry of the ellipse and then discuss the three laws.

Kepler’s 1st Law:

- The Sun is at one focus
- Nothing is located at the other focus
- Aphelion is the point farthest away from the Sun
  - The distance for aphelion is $a + c$
  - For a orbit around the Earth, this point is called the apogee
- Perihelion is the point nearest the Sun
  - The distance for perihelion is $a - e$
  - For an orbit around the Earth, this point is called the perigee

Notes About Ellipses

- $F_1$ and $F_2$ are each a focus of the ellipse ($r_1 + r_2$ constant)
- They are located a distance $c$ from the center
- $a$ is the semimajor axis
- $b$ is the semiminor axis
- The eccentricity of the ellipse is defined as $e = c/a$
- For a circle, $e = 0$
- The range of values of the eccentricity for ellipses is $0 < e < 1$

\[ e = c/a \]
Kepler’s First Law

- A circular orbit is a special case of the general elliptical orbits
- Is a direct result of the inverse square nature of the gravitational force
- Elliptical (and circular) orbits are allowed for bound objects
- A bound object repeatedly orbits the center
- An unbound object would pass by and not return
  » These objects could have paths that are parabolas ($e = 1$) and hyperbolas ($e > 1$)

Kepler found that the orbit of Mars was an ellipse, not a circle.

Kepler’s Second Law

The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.
Is a consequence of conservation of angular momentum
The force produces no torque, so angular momentum is conserved.

Central force (along radius) implies angular momentum conserved.

$$L = \vec{r} \times \vec{p} = m \vec{r} \times \vec{v} = \text{constant}$$

$$dA = \frac{1}{2} |\vec{r} \times d\vec{v}| = \frac{1}{2} |\vec{r} \times \vec{v}| dt = \frac{L}{2m} dt$$

$$\Rightarrow \frac{dA}{dt} = \frac{L}{2m} = \text{constant}$$

Kepler’s Third Law

Kepler had access to very good data from the astronomer Tycho Brahe in Prague. See table for today’s data.

After many years of work, Kepler found an intriguing correlation between the orbital periods and the length of the semimajor axis of orbits.

$$T^2 = Cr^3$$

Table 11-1: Mean orbital radii and orbital periods for the planets

<table>
<thead>
<tr>
<th>Planet</th>
<th>Mean Radius (in km)</th>
<th>Period (yr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>5,791</td>
<td>0.24</td>
</tr>
<tr>
<td>Venus</td>
<td>10,820</td>
<td>0.62</td>
</tr>
<tr>
<td>Earth</td>
<td>14,960</td>
<td>1.00</td>
</tr>
<tr>
<td>Mars</td>
<td>22,730</td>
<td>1.88</td>
</tr>
<tr>
<td>Jupiter</td>
<td>74,900</td>
<td>11.88</td>
</tr>
<tr>
<td>Saturn</td>
<td>1,427,000</td>
<td>29.5</td>
</tr>
<tr>
<td>Uranus</td>
<td>2,550</td>
<td>84</td>
</tr>
<tr>
<td>Neptune</td>
<td>4,924</td>
<td>102</td>
</tr>
<tr>
<td>Pluto</td>
<td>2,370</td>
<td>248</td>
</tr>
</tbody>
</table>
Of the satellites shown revolving around Earth, the one with the greatest speed is

- 1
- 2
- 3
- 4
- 5

The constant-area law.

The orbits of two planets orbiting a star are shown. The semimajor axis of planet A is twice that of planet B. If the period of planet B is $T_B$, the period of planet A is

A. $2T_B$
B. $2\sqrt{2}T_B$
C. $3T_B$
D. $\sqrt{3}T_B$
E. $4T_B$

Newton’s Universal Law of Gravity

Newton’s Law of Gravity

- Newton’s law of gravity will provide a physical theory of Kepler’s laws.

$F = G \frac{mM}{r^2}$
The Cavendish experiment

- G was first measured by Henry Cavendish in 1798.
- The apparatus shown here allowed the attractive force between two spheres to cause the rod to rotate.
- The mirror amplifies the motion.
- It was repeated for various masses.

Measuring G

\[ G = \frac{mM}{r^2} \]

Gravitational and inertial mass

- Gravitation is a force that acts on the gravitational mass:
  \[ F_g = G \frac{m_1 m_2}{r^2} \]
  (the masses are the source)
- Newton’s Law of motion acts on the inertial mass:
  \[ F = m_1 a \]
- In principle, they are not necessarily related, that the gravitational mass \( m_g \) is not the same as \( m_i \).
  But they are, up to the current experimental accuracy: Equivalence principle: gravity is equivalent to acceleration.
Kepler’s Third Law derived from Newton’s Law

Easily derived for a circular orbit

Centripetal force = gravitational force

\[ F_{\text{centripetal}} = F_{\text{gravitational}} \]

\[ m \frac{v^2}{R} = \frac{GMm}{R^2} \]

\[ m \omega^2 = \frac{GM}{R^2} \]

\[ \omega = \frac{2\pi}{T} \]

\[ \frac{GM}{R^2} \frac{4\pi^2}{T^2} = \text{const.} \]

Kepler’s Third Law

\[ T^2 = \frac{4\pi^2}{GM} R^3 \]

Extension for elliptical orbits, (Without proof \( R \rightarrow a \))

Kepler’s Third Law

\[ T^2 = \frac{4\pi^2}{GM} a^3 \]

Where \( a \) is the semimajor axis

Example, Mass of the Sun

- Using the distance between the Earth and the Sun, and the period of the Earth’s orbit, Kepler’s Third Law can be used to find the mass of the Sun

\[ \frac{GM_{\text{Sun}}}{R^2} = \frac{4\pi^2}{T^2} \]

\[ M_{\text{Sun}} = \frac{4\pi^2 R^3}{GT^2} \]

\[ M_{\text{Sun}} = 1.99 \times 10^{30} \text{ kg} \]

- Similarly, the mass of any object being orbited can be found if you know information about objects orbiting it

A woman whose weight on Earth is 500 N is lifted to a height of two Earth radii above the surface of Earth. Her weight

A. decreases to one-half of the original amount.
B. decreases to one-quarter of the original amount.
C. does not change.
D. decreases to one-third of the original amount.
E. decreases to one-ninth of the original amount.
From work to gravitational potential energy.

In the example before, it does not matter on what path the person is elevated to 2 Earth radii above. Only the final height (or distance) matters for the total amount of work performed.

Potential Energy

Work done to bring mass \( m \) from initial to final position.

\[
PE = -W = - \int_{r_i}^{r_f} \left( -\frac{G m M}{r} \right) dr = \frac{1}{2} \left( \frac{1}{r_f} - \frac{1}{r_i} \right) \frac{G m M}{r^2} \]

\[PE(\infty) = 0\]

Zero point is arbitrary. Choose zero at infinity.

\[
U(r) = -\frac{G M m}{r} \]

Force \((1.2) = -\frac{G m M}{r^2} \quad G = 6.67 \times 10^{-11} \text{ N m}^2 / \text{kg}^2\)