Chapter 6
Work and Energy

Work (dot product)
Work by gravity, by spring
Kinetic energy, power
Work-kinetic energy theorem
C.M. system

Definition of Work: Constant Force

**Ingredients:** Force \( \mathbf{F} \), displacement \( \Delta \mathbf{r} \)

Work, \( W \), of a constant force \( \mathbf{F} \)
acting through a displacement \( \Delta \mathbf{r} \):

\[
W = \mathbf{F} \cdot \Delta \mathbf{r} = F \Delta r \cos \theta = F_x \Delta x
\]

*Dot Product*  Work is a *scalar*.

Units are Newton meter: Joule = N x m

More on “dot product” (or scalar product)

\[
\mathbf{\hat{a}} \cdot \mathbf{\hat{b}} = AB \cos \theta \quad \text{(definition)} \quad \mathbf{\hat{a}} \cdot \mathbf{\hat{b}} = A \hat{a}_x B \hat{b}_x + A \hat{a}_y B \hat{b}_y
\]

Definition:

\[
\mathbf{\hat{a}} \cdot \mathbf{\hat{b}} = ab \cos \theta
\]

\[
= a \mathbf{\hat{b}} \cdot (\cos \theta) = ab
\]

\[
= q(\cos \theta) = ab
\]

Some properties:

\[
\mathbf{\hat{a}} \cdot \mathbf{\hat{b}} = \mathbf{\hat{b}} \cdot \mathbf{\hat{a}}
\]

\[
q(\mathbf{\hat{a}} \cdot \mathbf{\hat{b}}) = q(\mathbf{\hat{b}} \cdot \mathbf{\hat{a}})
\]

\[
\mathbf{\hat{a}} \cdot (\mathbf{\hat{b}} + \mathbf{\hat{c}}) = (\mathbf{\hat{a}} \cdot \mathbf{\hat{b}}) + (\mathbf{\hat{a}} \cdot \mathbf{\hat{c}})
\]

\( q \) is a scalar

\( \mathbf{\hat{a}} \) is a vector

The dot product of perpendicular vectors is 0 !!
Examples of dot products

\[
\begin{align*}
\mathbf{i} \cdot \mathbf{j} &= 0 \\
\mathbf{i} \cdot \mathbf{k} &= 0 \\
\mathbf{j} \cdot \mathbf{k} &= 0 \\
\mathbf{i} \cdot \mathbf{i} &= 1 \\
\mathbf{j} \cdot \mathbf{j} &= 1 \\
\mathbf{k} \cdot \mathbf{k} &= 1
\end{align*}
\]

Suppose \( \mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \), then

\[
\mathbf{a} \cdot \mathbf{b} = (a_x, a_y, a_z) \cdot (b_x, b_y, b_z) = a_x b_x + a_y b_y + a_z b_z
\]

Which of the statements below is correct?

A. The scalar product of two vectors can be negative.
B. \( \mathbf{A} \cdot \mathbf{B} = c(\mathbf{B} \cdot \mathbf{A}) \), where \( c \) is a constant.
C. The scalar product can be non-zero even if two of the three components of the two vectors are equal to zero.
D. If \( \mathbf{A} = \mathbf{B} + \mathbf{C} \), then \( \mathbf{D} \cdot \mathbf{A} = \mathbf{D} \cdot \mathbf{B} + \mathbf{D} \cdot \mathbf{C} \)
E. All of the above statements are correct.

More properties of dot products

- Components:
  \[
  \mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} = (a_x, a_y, a_z) = (\mathbf{i} \cdot \mathbf{a} \cdot \mathbf{j} \cdot \mathbf{a} \cdot \mathbf{k})
  \]

- Derivatives:
  \[
  \frac{d}{dt} (\mathbf{a} \cdot \mathbf{b}) = \frac{d\mathbf{a}}{dt} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{d\mathbf{b}}{dt}
  \]

- Apply to velocity
  \[
  \frac{d}{dt} \mathbf{v} = \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v}) = \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} + \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = 2\mathbf{v} \cdot \frac{d\mathbf{v}}{dt}
  \]

So if \( \mathbf{v} \) is constant

(like for uniform circular motion):

\[
\frac{d}{dt} \mathbf{v} = 2\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = 0
\]

What about multiple forces?

Suppose \( \mathbf{F}_{\text{net}} = \mathbf{F}_1 + \mathbf{F}_2 \) and the displacement is \( \Delta \mathbf{r} \).

The work done by each force is:

\[
\begin{align*}
W_1 &= \mathbf{F}_1 \cdot \Delta \mathbf{r} \\
W_2 &= \mathbf{F}_2 \cdot \Delta \mathbf{r} \\
W_{\text{net}} &= \mathbf{F}_{\text{net}} \cdot \Delta \mathbf{r} \\
&= (\mathbf{F}_1 + \mathbf{F}_2) \cdot \Delta \mathbf{r} \\
&= \mathbf{F}_1 \cdot \Delta \mathbf{r} + \mathbf{F}_2 \cdot \Delta \mathbf{r}
\end{align*}
\]

\[
W_{\text{net}} = \mathbf{F}_{\text{net}} \cdot \Delta \mathbf{r}
\]

It's the total force that matters.
You are towing a car up a hill with constant velocity. The work done on the car by the normal force is:
1. positive
2. negative
3. zero
   - correct

Normal force is perpendicular to displacement, \( \cos \theta = 0 \).

You are towing a car up a hill with constant velocity. The work done on the car by the gravitational force is:
1. positive
2. negative
3. zero

There is a non-zero component of gravitational force pointing opposite the direction of motion.

You are towing a car up a hill with constant velocity. The work done on the car by the tension force is:
1. positive
2. negative
3. zero

T is pointing in the direction of motion - therefore, work done by this force is positive.

You are towing a car up a hill with constant velocity. The total work done on the car by all forces is:
1. positive
2. negative
3. zero

Constant velocity implies that there is no net force acting on the car, so there is no work being done overall.
Lifting an object:
A 3000-kg truck is to be loaded onto a ship by a crane that exerts an upward force of 31 kN on the truck. This force, which is strong enough to overcome the gravitational force and keep the truck moving upward, is applied over a distance of 2.0 m. Find
(a) the work done on the truck by the crane,
(b) the work done on the truck by gravity,
(c) the net work done on the truck.
(a) the work done by the crane:
\[ W_c = F_c d = 31 \text{ kN} \times 2 \text{ m} = 62 \text{ kJ} \]
(b) the work done by gravity:
\[ W_g = F_g d = (-31 \text{ kN}) \times 2 \text{ m} = -62 \text{ kJ} \]
(c) the net work done on the truck:
\[ W = W_c + W_g = 0 \]
Energy
Energy is that quality of a substance or object which "causes something to happen"; or "capability of exerting forces"; or "ability to do work"...

The vagueness of the definition is due to the fact that energy can result in many effects.

- Electrical Energy
- Chemical Energy
- Mechanical Energy
- Nuclear Energy

It is convertible into other forms without loss (i.e., it is conserved).

Kinetic Energy
The "energy of motion".

Work done on the object increases its energy, -- by how much? (i.e., how to calculate the value?)

\[ W = F \cdot d = ma \cdot d = \frac{1}{2}mv^2 - \frac{1}{2}m_0 v_0^2 \]

Definition of Kinetic Energy
Kinetic energy (K.E.) of a particle of mass \( m \) moving with speed \( v \) is defined as

\[ K.E. = \frac{1}{2}mv^2 \]

Kinetic energy is a useful concept because of the Work/Kinetic Energy theorem, which relates the work done on an object to the change in kinetic energy.

\[ W_{ext} = \Delta K \]

WORK = KINETIC-ENERGY THEOREM

A falling object
What is the speed of an object that starts at rest and then falls a vertical distance \( H \)?

Work done by gravitational force

\[ W_G = F \cdot \Delta r = mgH \]

Work/Kinetic Energy Theorem:

\[ W_G = \frac{1}{2}mv^2 \]

\[ \Rightarrow v = \sqrt{2gH} \]
A skier of mass 50 kg is moving at speed 10 m/s at point \( P_1 \) down a ski slope with negligible friction. What is the skier’s kinetic energy when she is at point \( P_2 \), 20 m below \( P_1 \)?

\[
\text{KE}_{f} - \text{KE}_{i} = W_g = mgH
\]

\[
\text{KE}_{f} = \frac{1}{2} \times 50 \times 10^2 + 50 \times 9.8 \times 20 = 2500 \text{ J} + 9800 \text{ J}
\]

A. 2500 J  
B. 9800 J  
C. 12300 J  
D. 13100 J  
E. 15000 J

Energy and Newton’s Laws

The importance of mechanical energy in classical mechanics is a consequence of Newton’s laws.

The concept of energy turns out to be even more general than Newton’s Laws.

Problem: Work and Energy

Two blocks have masses \( m_1 \) and \( m_2 \), where \( m_1 > m_2 \). They are sliding on a frictionless floor and have the same kinetic energy when they encounter a long rough stretch with \( \mu > 0 \), which slows them down to a stop.

Which one will go farther before stopping?

(A) \( m_1 \)  
(B) \( m_2 \)  
(C) They will go the same distance

Solution next page

Problem: Work and Energy (Solution)

The net work done to stop a box is \(-FD = -\mu mgD\).

The work-kinetic energy theorem says that for any object,

\[
W_{net} = \Delta K \quad \text{so} \quad W_{net} = K_f - K_i.
\]

Since the boxes start out with the same kinetic energy, we have

\[
\mu m_1 g D_1 = \mu m_2 g D_2 \quad \text{and} \quad D_2/D_1 = m_2/m_1.
\]

Since \( m_1 > m_2 \), we must have \( D_2 > D_1 \).
Problem:

The magnitude of the single force acting on a particle of mass \( m \) is given by \( F = bx^2 \), where \( b \) is a constant. The particle starts from rest at \( x=0 \). After it travels a distance \( L \), determine its (a) kinetic energy and (b) speed.

work done by force:

\[
W = \int_0^L F \cdot dx = \int_0^L bx^2 \cdot dx = \frac{bL^3}{3}
\]

So (a) kinetic energy is \( K = \frac{bL^3}{3} \)

(b) Since \( \frac{1}{2}mv^2 = \frac{bL^3}{3} \),

\[
v = \sqrt{\frac{2bL^3}{3m}}
\]

Power:

P: Work done by unit time.

\[
\delta W = \vec{F} \cdot \delta \vec{r} = \vec{F} \cdot \vec{v} \cdot dt
\]

\[
P = \frac{dW}{dt} = \vec{F} \cdot \vec{v}
\]

Units: Watt

\( 1 \text{ W} = 1 \text{ J/s} \)

\( 1 \text{ hp} = 550 \text{ ft} \cdot \text{lb/s} \approx 746 \text{ W} \)

Question

A sports car accelerates from zero to 30 mph in 1.5 s. How long does it take for it to accelerate from zero to 60 mph, assuming the power of the engine to be independent of velocity and neglecting friction?

Power is constant, so

\[
\frac{d}{dt} \left( \frac{1}{2}mv^2 \right) = C, \text{ a constant.}
\]

Integrating with respect to time, and noting that the initial velocity is zero, one gets \( \frac{1}{2}mv^2 = Ct \). So getting to twice the speed takes 4 times as long, and the time to reach 60 mph is \( 4 \times 1.5 = 6 \text{ s.} \)

If a fighter jet doubles its speed, by what factor should the power from the engine change? (Assume that the drag force on the plane is proportional to the square of the plane’s speed.)

A. by half
B. unchanged
C. doubled
D. quadrupled
E. 8 times

Magnitude of power is \( Fv \). When the velocity \( v \) is doubled, the drag force goes up by a factor of 4, and \( Fv \) goes up by a factor of 8.
Recall: Center of Mass

Definition of the center of mass:

\[ \vec{R}_{cm} = \sum_j m_j \vec{r}_j \Rightarrow \vec{a}_{cm} = \frac{\sum_j m_j \vec{a}_j}{\sum_j m_j} . \]

Because \[ \vec{F}_i = m_i \vec{a}_i , \]

\[ \sum_i \vec{F}_i = \sum_i m_i \vec{a}_i = \left( \sum_i m_i \right) \left( \frac{\sum_j m_j \vec{a}_j}{\sum_j m_j} \right) = M_{cm} \vec{a}_{cm} . \]

Center-Of-Mass Work

For systems of particles that are not all moving at the same velocity, there is a work-kinetic energy relation for the center of mass.

So

\[ \vec{F}_{\text{net,ext}} \cdot \vec{v}_{cm} = M_{cm} \vec{a}_{cm} = \frac{d}{dt} \left( \frac{1}{2} M_{cm} v_{cm}^2 \right) = \frac{dK_{\text{trans}}}{dt} \]

where \( K_{\text{trans}} = \frac{1}{2} m v_{cm}^2 \) is the translational kinetic energy.

\[ \int_{x_{\text{initial}}}^{x_{\text{fin}}} \vec{F}_{\text{net,ext}} \cdot dx_{cm} = \Delta K_{\text{trans}}. \]

Net work done on collection of objects \( \equiv \) change in translational kinetic energy of system.