The radial Hamiltonian for the Hydrogen atom with orbital angular momentum quantum number \( l \) is

\[
H_l = -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{\hbar^2 l(l + 1)}{2\mu r^2} - \frac{e^2}{r}
\]

We wish to find the energies \( E_{n_r,l} \) and corresponding wave functions \( P_{n_r,l}(r) \). It is convenient to define a radial quantum number \( n_r \) which is equal to the number of nodes in the wavefunction.

For small \( r \), the wavefunctions must be proportional to \( r^{l+1} \). At large \( r \), they must decay as \( e^{-\kappa r} \), where \( \kappa = \sqrt{-2\mu E/\hbar^2} \). The lowest energy level for any \( l \) must have no zero crossings. Thus an educated guess for the wave function for the lowest energy level for a given \( l \) is

\[
P_{0,l}(r) = r^{l+1}e^{-\frac{r}{a}}
\]

If we plug this into the Schrodinger equation, as in the accompanying Mathematica notebook, we find

\[
\varepsilon \frac{r}{a} (2 a (1 + 1) \hbar^2 + r \hbar^2 + 2 a^2 (e^2 + r \varepsilon) \mu) = 0
\]

For this to hold at any \( r \), each power of \( r \) in the numerator must individually vanish. Thus

\[-2 a (1 + 1) \hbar^2 + 2 a^2 e^2 \mu = 0, \ r \hbar^2 + 2 a^2 r \varepsilon \mu = 0\]

which has the solution

\[a = (l + 1)a_B, \quad E_{0,l} = -\frac{Ry}{(l + 1)^2}\]

where the Bohr radius is \( a_B = \hbar^2/\mu e^2 \) and the Rydberg constant is \( Ry = \mu e^4/2\hbar^2 \).

We now have the lowest energy level and its wavefunction for each \( l \).

Now we want to find the excited state energy levels. We will use a trick similar to the raising and lowering operators for the harmonic oscillator. We define a "superpotential"

\[
W_l(r) = \frac{e^2 \mu}{(l + 1)\hbar^2} - \frac{l + 1}{r}
\]

and we define operators

\[
A_l = \frac{d}{dr} + W_l
\]

\[
A_l^\dagger = -\frac{d}{dr} + W_l
\]

then we can show (see Mathematica)
\[
\frac{\hbar^2}{2\mu} A_l^+ A_l = H_l - E_{0l} \\
\frac{\hbar^2}{2\mu} A_l A_l^+ = H_{l+1} - E_{0l}
\]  

5) Now we wish to show that \( A^+ P_{n_r,l+1} = P_{n_r+1,l} \) and \( E_{n_r+1,l} = E_{n_r,l+1} \). The creation operator adds a node to the wavefunction and reduces its angular momentum, but keeps the energy the same.

Let us operate \( H_l \) on \( A^+ P_{n_r,l+1} \):

\[
H_l A_l^+ P_{n_r,l+1} = \left( \frac{\hbar^2}{2\mu} A_l^+ A_l + E_{0l} \right) A_l^+ P_{n_r,l+1} \\
= A_l^+ \left( \frac{\hbar^2}{2\mu} A_l A_l^+ + E_{0l} \right) P_{n_r,l+1} \\
= A_l^+ H_{l+1} P_{n_r,l+1} = E_{n_r,l+1} A_l^+ P_{n_r,l+1}
\]  

6) Therefore \( A^+ P_{n_r,l+1} \) is an eigenfunction of \( H_l \) with eigenvalue \( E_{n_r,l+1} \). We therefore have

\[
E_{n_r,l} = E_{0,n_r+l} = - \frac{Ry}{(n_r + l + 1)^2}
\]  

7) The action of the operator \( A_l^+ \) gives us a new eigenstate of one less \( l \), and one greater \( n_r \). Defining the principal quantum number \( n = n_r + l + 1 \), it follows that the energy levels of the hydrogen atom energy levels are

\[
E_{nl} = - \frac{Ry}{n^2}
\]  

8) and the eigenfunctions can be generated by successively operating on \( P_{0n}(r) \):

\[
P_{nl}(r) = A_l^+ ... A_{n-3}^+ A_{n-2}^+ P_{0n-1}(r)
\]  

9) where \( A_l^+ \) is the lowering operator corresponding to superpotential \( W_l \).