Phys 448 HW 8

1) BD 7.1
2) BD 7.2
3) BD 7.4
4) BD 7.6 (worth double)
5) Take the dye Hamiltonian of BD problem 6.2. Consider the operator \( \rho = |1\rangle\langle 2| + |2\rangle\langle 3| + |3\rangle\langle 1| \). Is \( \rho \) Hermitian? Is it an observable? Does \( \rho \) commute with the Hamiltonian? Find the eigenvectors of \( \rho \). Show that they are eigenvectors of H.

6) Now consider the operator \( B = ib(\rho - \rho^\dagger) \). Is \( B \) Hermitian? Is it an observable? Does it commute with H? Do H and B constitute a CSCO? Find the eigenvectors and eigenvalues of \( B \). Show that the eigenvectors of \( B \) are eigenvectors of H. Now that you know the eigenvalues of \( B \) and H, what are the energies of the Hamiltonian \( H' = H + B \)?

7) An atom has two states \( |g\rangle \) and \( |e\rangle \), resonantly (i.e. the frequency of the light is equal to the Bohr frequency between the two states) coupled by light, giving an effective interaction

\[ V = \frac{\hbar \varepsilon}{2} |g\rangle\langle e| + \frac{\hbar \varepsilon}{2} |e\rangle\langle g| \]

Find the eigenvectors and eigenvalues of the light-atom system. Find the wavefunction as a function of time, for an arbitrary initial wavefunction, that is find the time evolution matrix \( U(t) \) such that \( \psi(t) = U(t)\psi(0) \). Show that after a time \( \varepsilon t = \pi \) (a “\( \pi \)” pulse) an initial state \( \psi(0) = |g\rangle \) evolves to \( \psi = -i|e\rangle \) and that after a 2\( \pi \) pulse \( \psi = -|g\rangle \). How long must you wait for \( \psi(t) = \psi(0) \)?

8) With the light turned off, the effective atomic Hamiltonian is

\[ H_0 = \hbar(\omega_0 - \omega)|e\rangle\langle e| \]

Find the evolution matrix \( U_0(t) \). Suppose the atom starts in state \( g \). First a \( \pi/2 \) pulse is applied. Then the light is turned off for a time \( T \). Finally, another \( \pi/2 \) pulse is applied. After this “Ramsey” sequence, what is the probability of finding the atom in state \( e \)? Plot it as a function of \( T \).

9) Suppose now that the above happens to a set of atoms for differing times \( T \). This might arise, for example, because the atoms move through a light-free region of space with differing velocities. Suppose the distribution of times is Gaussian, with mean \( \bar{T} \) and standard deviation \( \sigma_T \). Find the transition probability as a function of time. Plot your result for
\((\omega_0 - \omega)T = 200\pi\), and \(\sigma_T T = 4\). How might you use this phenomenon to measure \((\omega_0 - \omega)\) very precisely?

Decoding Problem 3:
“An atom has two states \(|g\rangle\) and \(|e\rangle\).”

This means there are going to be two states of interest in this problem. The atom by itself will have the Hamiltonian

\[
\begin{pmatrix}
E_g & 0 \\
0 & E_e
\end{pmatrix}
\]

“resonantly coupled by light”

This means, first of all, that there is an oscillating electromagnetic field present. The Hamiltonian of the system “atom + light” is time dependent. However, what we showed in class is that the time dependent “atom + light” system is equivalent to a time-independent system where the atom energies are shifted by \(\pm \hbar \omega / 2\), and where the \(\cos \omega t\) in the interaction between the atom and the light is replaced by \(1/2\). If for the time being we ignore the interaction part, the Hamiltonian of the system “atom + light” is therefore

\[
H_{AL} = \begin{pmatrix}
E_g + \hbar \omega / 2 & 0 \\
0 & E_e - \hbar \omega / 2
\end{pmatrix}
= \begin{pmatrix}
0 & 0 \\
0 & E_e - E_g - \hbar \omega
\end{pmatrix}
\]

The second piece of information in the above phrase is that the light is resonant. This means that the Bohr condition is met, \(E_e - E_g = \hbar \omega\). Thus

\[
H_{AL} = E_g + \hbar \omega / 2 + \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\]

“giving an effective interaction \(V = \frac{\hbar \varepsilon}{2} |g\rangle\langle e| + \frac{\hbar \varepsilon}{2} |e\rangle\langle g|\).”

This is telling about the interaction between the atom and the light. Since the \(\cos \omega t\) is not mentioned, the interaction is being described in the time-independent representation. Thus the Hamiltonian for this problem is

\[H = H_{AL} + V\]