Physics 202, Lecture 3

Today’s Topics

- Calculate Electric Field With Superposition (Direct Sum/Integral of Coulomb’s Law)
- Calculate Electric Field With Gauss’s Law
  - Gauss’s Law
  - Examples

Expected from preview:
Calculate E with continuous charge.
Surface, closed surface, surface integral, flux, the Gauss’s Law.
Review: Electric Field and Electric Force

Electric Field is a form of matter. It carries energy (later in the semester)

\[ \vec{F} = \vec{E} q = K_e \frac{q_0}{r^2} \hat{r} \]

\[ \vec{E} = K_e \frac{q_0}{r^2} \hat{r} \]

$q_0$: source charge
$E$: field by $q_0$
$q$: test charge
$F = qE$ force on $q$ by $E$
How to Calculate Electric Field?

- Single point-like:
  \[ \vec{E} = k_e \frac{q_0}{r^2} \hat{r} \]

- Multiple charges:
  \[ \vec{E} = k_e \sum \frac{q_i}{r_i^2} \hat{r}_i \]
  (superposition principle)

- Continuous Charge Distribution:
  \[ \vec{E} = k_e \lim_{\Delta q \to 0} \sum \frac{\Delta q_i}{r_i^2} \hat{r}_i = k_e \int \frac{dq}{r^2} \hat{r} \]

Note: For now, we assume charges are not moving.
(electrostatic)
Example: Charged Rod

- A uniformly charged rod of length $L$ has a total charge $Q$, find the electric field:
  - at point A \( \Rightarrow \) answer: \( E_x = -k_e Q/(a(L+a)) \), \( E_y = 0 \) (see board)
  - at point B \( \Rightarrow \) answer: \( E_y = 2k_e Q/(Lb) \cdot \sin \theta_0 \), \( E_x = 0 \) \( \tan \theta_0 = \frac{1}{2} L/b \) (see board, show method only)
  - at an arbitrary point C. (see board, conceptual only).
Example: Uniformly Charged Sphere

- A uniformly charged sphere has a radius \( a \) and total charge \( Q \), find the electric field outside and inside the sphere.

- Solution:

Don’t take notes of my solution: I AM FOOLING ARUOND! It is quite complicated with the superposition method!

⇒ Gauss’s Law to the rescue!
Electric Flux

- The **electric flux** through a **surface element** is defined as the dot product of the **electric field** and the **surface area vector**:
  \[ \Delta \Phi_E = \mathbf{E} \cdot \Delta \mathbf{A} = E \Delta A \cos \theta \]

- The net electric flux through a **closed surface**

\[ \Phi_E \equiv \oint \mathbf{E} \cdot d\mathbf{A} \]
The Gauss’ s Law

- The Gauss’ s Law: The net electric flux through any closed surface (also called Gaussian surface) equals to the total charge enclosed inside the closed surface divided by the free space permittivity.

\[ \Phi_E \equiv \oint E \cdot dA = \sum \frac{q_{in}}{\varepsilon_0} \]

- \( q_{in} \): all \( q \)'s enclosed regardless of positions
- \( \Phi_E \): flux
- \( \varepsilon_0 \): permittivity constant
- \( \frac{1}{4\pi\varepsilon_0} = k_e \)
- Gaussian surface (any shape)
Trivia Quiz 1

- Compare electric fluxes through closed surfaces $s_1, s_2, s_3$:
  1. $\Phi_{s1} > \Phi_{s2} > \Phi_{s3}$
  2. $\Phi_{s1} = \Phi_{s2} = \Phi_{s3}$
  3. $\Phi_{s1} < \Phi_{s2} < \Phi_{s3}$
Trivia Quiz 2

What is the electric flux through closed surface S?

1. $\Phi = 0$

2. $\Phi = \frac{q_1 + q_2 + q_3 + q_4 + q_5}{\varepsilon_0}$

3. $\Phi = \frac{q_1 + q_2 + q_3}{\varepsilon_0}$
Uniformly Charged Sphere Again

- Solution using Gauss’ s Law:

\[ \int \vec{E} \cdot d\vec{A} = \frac{1}{\varepsilon_0} q_{in} \]

The setting is highly symmetrical

\( \Rightarrow \) Gaussian surface will be concentric sphere of radius \( r \).

How to evaluate \( \int \vec{E} \cdot d\vec{A} \)?

Note the symmetry:

\( \Rightarrow \) Direction of \( E \): Radial
\( \Rightarrow \) Magnitude of \( E \): Same in all direction

\[ \int \vec{E} \cdot d\vec{A} = \int E dA = E \int dA \]

\[ = EA = 4\pi r^2 E \]
Uniformly Charged Sphere: Details

\[ \int \vec{E} \cdot d\vec{A} = 4\pi r^2 E = \frac{1}{\varepsilon_0} q_{in} \]

where: \( q_{in} = \begin{cases} \frac{Q}{r^3} & \text{if } r > a \\ \frac{r^3 Q}{a^3} & \text{if } r < a \end{cases} \)

\[ E = \frac{1}{4\pi r^2 \varepsilon_0} q_{in} \]
Uniform Charge Sphere: Final Solution

Note: This has the same form as the point charge

\[ E = \frac{k_e Q}{a^3} r \]

inside outside

\[ E = \frac{k_e Q}{r^2} \]
Procedure to Use Gauss’ s Law

\[ \oint E \cdot dA = \frac{q_{in}}{\varepsilon_0} \]

- General principle:
  Gauss’ s law is valid for any charge distributions, but practically it is useful only in limited situations where the charge distribution is highly symmetric.

- Procedure:
  1. Draw a Gaussian surface passing the field point concerned. Observe symmetry so that the surface integral is trivial.
     - Direction of E: Either perpendicular or parallel to the surface.
     - Magnitude of E: The same (or be zero) on the surfaces
  2. Evaluate the surface integral using arguments of symmetry. And Equate the surface integral to \( q_{in}/\varepsilon_0 \) and solve for E.
Three Common Symmetric Cases

- **Spherical**
  (point Q, uniform sphere, shell)

- **Cylindrical**
  (infinite line/cylinder of Q)

- **Planar**
  (infinite sheet of Q)

The above symmetric settings give very predictable E

→ **Direction:** Normal to surfaces of same symmetry

→ **Magnitude:** Same across surface (of same symmetry)

\[ \oint \vec{E} \cdot d\vec{A} = E \cdot A \]
Another Example: Thin Spherical Shell

- Find the E field inside/outside a uniformly charged thin sphere.

Solution: Exercise with your TAs.
## Typical Electric Field Obtainable By Gauss’s Law

<table>
<thead>
<tr>
<th>Charge Distribution</th>
<th>Electric Field</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>Insulating sphere of radius $R$, uniform charge density, and total charge $Q$</td>
<td>$k_e \begin{cases} \frac{Q}{r^2} &amp; r &gt; R \ \frac{Q}{R^2} r &amp; r &lt; R \end{cases}$</td>
<td>$r &gt; R$</td>
</tr>
<tr>
<td>Thin spherical shell of radius $R$ and total charge $Q$</td>
<td>$k_e \begin{cases} \frac{Q}{r^2} &amp; r &gt; R \ 0 &amp; r &lt; R \end{cases}$</td>
<td>$r &lt; R$</td>
</tr>
<tr>
<td>Line charge of infinite length and charge per unit length $\lambda$</td>
<td>$2k_e \frac{\lambda}{r}$</td>
<td>Outside the line</td>
</tr>
<tr>
<td>Infinite charged plane having surface charge density $\sigma$</td>
<td>$\frac{\sigma}{2\varepsilon_0}$</td>
<td>Everywhere outside the plane</td>
</tr>
<tr>
<td>Conductor having surface charge density $\sigma$</td>
<td>$\begin{cases} \frac{\sigma}{\varepsilon_0} &amp; \text{Just outside the conductor} \ 0 &amp; \text{Inside the conductor} \end{cases}$</td>
<td></td>
</tr>
</tbody>
</table>
One More Exercise On Gauss’ s Law

- Two charges +2Q and −Q are placed at locations shown. Find the electric field at point P.

- Solution:
  1. Draw a Gaussian surface passing P
  2. Apply Gauss’ s law:
      \[ \oint E \cdot dA = \frac{q_{\text{in}}}{\varepsilon_0} \]
  3. \( q_{\text{in}} = +2Q + (-Q) = Q \)
  4. Surface integral:
      \[ \oint E \cdot dA = 4\pi r^2 E \]
  5. \( \Rightarrow E = \frac{1}{4\pi\varepsilon_0} \left( \frac{Q}{r^2} \right) \)

Is this correct? No! Which step is wrong? step 4