There is a specialized method, called the Numerov method, for numerically integrating differential equations of the form
\[ \psi'' = f(x)\psi \]

The method, which you can read about on Wikipedia, approximately calculates \( \psi \) at a set of \( N \) discrete positions \( x_i \). Specifically it is a formula for \( \psi_{i+1} \) in terms of \( \psi_i, \psi_{i-1}, f_{i+1}, f_i, \) and \( f_{i-1} \). Your job is write a Mathematica program that will find the eigenvalues and eigenfunctions of the Schrodinger equation for a potential \( V(x) \) that has a characteristic length scale \( a \). Here’s how to proceed.

First, find \( f(x) \), then write it in terms of an appropriately scaled length \( s=x/a \) and a corresponding scaled energy variable \( \epsilon \), so that \( \hbar \) and \( m \) are eliminated from the equations. Next, rewrite the Numerov formula in the following manner. Consider \( \psi \) to be a length \( N \) vector whose elements are the \( \psi_i \). The Numerov formula can then be rewritten in the form
\[ A\psi + BU\psi = \epsilon B\psi \]

where \( A \) and \( B \) are \( N \times N \) matrices and \( U \) is an \( N \times N \) diagonal matrix. Multiply by \( B^{-1} \) to put the Numerov formula in eigenvalue form
\[ K\psi + U\psi = \epsilon \psi \]

In this form, you can numerically find the eigenvalues and eigenvectors of the Schrodinger equation for, in principle, any potential.

As a test case, find the lowest 20 eigenvalues and eigenvectors of the simple harmonic oscillator. Your results should be accurate to better than 0.1% in the energy. You will need to adjust \( N \) and the distance \( \delta s \) between points to get sufficient accuracy. Calculate the uncertainty product for the 17\(^{th}\) eigenvector and compare to theory. When you get this working, show me your program and I will give you a new potential to solve using it.

Write a short paper describing your theory and calculations. The student with the best solution will be invited to be a coauthor on a short paper on this topic to be submitted to the American Journal of Physics.

The paper and program are due Dec. 2.