Today's Topics

- Capacitance (Ch. 24.1-24.3)
  - Calculating capacitance
  - Combinations of capacitors: series and parallel
  - Energy stored in capacitors (24.4)

Capacitance

Capacitor: two (spatially separated) conductors, charged to +Q and -Q, with constant potential difference $\Delta V$

- Stores electrical energy (by storing charge)

$$C \equiv \frac{Q}{\Delta V}$$

Units: Farad (F)

$$1 \text{ F} = 1 \frac{\text{C}}{\text{V}}$$

Capacitors

Battery: source of $\Delta V$

Charging a Capacitor

- Battery: source of $\Delta V$
- Uncharged
- Charging
- Charge flows until $Q = C \Delta V$

Calculating Capacitance

Capacitance is independent of charge, voltage on capacitor: it depends on geometry of the conductors.

Examples:
- Parallel conducting plates
- Concentric spherical conductors
- Concentric cylindrical conductors

Procedure: determine $\Delta V$ as a function of $Q$

Simple example: capacitance of sphere (imagine second conductor at infinity)

$$\Delta V = \frac{Q}{R}$$

$$C = \frac{Q}{kQ/R} = \frac{R}{k} = 4\pi\varepsilon_0 R$$

Example: Parallel Plate Capacitor

Consider two metallic parallel plates with area A, separation d:

Step 1. Use Gauss's law to get:

$$\oint E \cdot dA = +Q$$

$$\sigma = Q/A$$

$$E = \frac{\sigma}{\varepsilon_0}$$

Step 2. Get potential difference:

$$\Delta V = -\int E \cdot dl = \frac{Qd}{\varepsilon_0 A}$$

Step 3. Get Capacitance:

$$C = \frac{Q}{\Delta V} = \frac{\varepsilon_0 A}{d}$$
Consider two concentric conducting spherical shells, radii \(a, b\) with \(a < b\).

**Parallel Plate Capacitor**

\[ C = \frac{\varepsilon_0 A}{d} \]

**Cylindrical Capacitor**

Consider two concentric cylindrical conducting shells, radii \(a, b\) with \(a < b\).

**Spherical Capacitor**

Step 1. Get electric field:

Gauss’s Law: \( \vec{E} = \frac{kQ}{r^2} \) for \(a < r < b\)

Step 2. Get potential difference:

\[ \Delta V = -\int \vec{E} \cdot d\vec{l} = -\int \frac{kQ}{r^2} dr = kQ \ln \left( \frac{b}{a} \right) \]

Step 3:

\[ C = \frac{Q}{\Delta V} = \frac{ab}{k(b-a)} \]

**Calculating Capacitance**

- Given 4 concentric cylinders of radii \(a, b, c, d\) and charges \(Q_1, Q_2, Q_3, Q_4\):
  - Charge conservation: \(Q_4 = Q_3 + Q_2\)
  - \(Q_1 = Q_4\)

\[ V_0 = \int_a^b \frac{Q}{2 \pi \varepsilon_{0} r L} \, dr = \frac{Q}{2 \pi \varepsilon_{0} L} \left( \ln \left( \frac{b}{a} \right) + \ln \left( \frac{d}{c} \right) \right) \]

\[ C = \frac{Q}{V_0} = \frac{2 \pi \varepsilon_{0} L}{\ln \left( \frac{b}{a} \right) + \ln \left( \frac{d}{c} \right)} \]

Note: result of 2 cylindrical capacitors “in series” (next few slides)

**Combinations of Capacitors**

**Parallel combination**: \( \Delta V \) same

\[ \frac{1}{C_p} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots \]

**Series combination**: \( \Delta V \) same

\[ \frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots \]

**Equivalent Capacitance**

\[ C_p = C_1 + C_2 + C_3 + \ldots \]

\[ C_p \text{ always } > C_i \]

\[ C_s \text{ always } < C_i \]
Example: Connection of Capacitors

 Imagine three capacitors \( C_1 = C_2 = C_3 = C \) with the configuration shown by the left diagram:

\[
C_{12} = C_1 + C_2 = 2C \\
C_{23} = C_2 + C_3 = 2C \\
C_{13} = C_1 + C_3 = 2C \\
C_{sw} = \frac{2}{3} C
\]

Energy of a Capacitor

- How much energy is stored in a charged capacitor?
  - Calculate the work provided (usually by a battery) to charge a capacitor to \( q \):

\[
dW = V(q)dq = \frac{q^2}{2C} \]

- The total work \( W \) to charge to \( q \) is then given by:

\[
W = \frac{1}{2} CV^2
\]

Capacity Variables

- The total work to charge capacitor to \( q \) equals the energy \( U \) stored in the capacitor:

\[
U = \frac{1}{2} C \int dq = \frac{q^2}{2C}
\]

- In terms of the voltage \( V \):

\[
U = \frac{1}{2} CV^2
\]

You can do one of two things to a capacitor:
- hook it up to a battery \( V \) specify \( q \) and \( V \) follows \( Q = CV \)
- put some charge on it \( q \) specify \( Q \) and \( V \) follows \( V = \frac{Q}{C} \)

Example (I)

- Suppose the capacitor shown here is charged to \( Q \). The battery is then disconnected.
- Now suppose the plates are pulled further apart to a final separation \( d_f \).
- How do the quantities \( Q, C, E, F, U \) change?
  - \( Q \): remains the same, no way for charge to leave.
  - \( C \): decreases, capacitance depends on geometry
  - \( E \): remains the same... depends only on charge density.
  - \( F \): increases, since \( C \), but \( Q \) remains same (or \( d_f \) but \( E \) the same)
  - \( U \): increases, add energy to system by separating
- How much do these quantities change?... See board.

Answers:

\[
C_f = \frac{d}{d_i} C \\
V_f = \frac{d}{d_i} V \\
U_f = \frac{d}{d_i} U
\]

Example (II)

- Suppose the battery \( V \) is kept attached to the capacitor.
- Again pull the plates apart from \( d \) to \( d_f \).
- Now what changes?
  - \( C \): decreases (capacitance depends only on geometry)
  - \( V \): must stay the same - the battery forces it to be \( V \)
  - \( Q \): must decrease, \( Q = CV \) charge flows off the plate
  - \( E \): must decrease \( (E = \frac{1}{d} \rightarrow \frac{d}{d_f} \) E)
  - \( U \): must decrease \( (U = \frac{Q^2}{C}) \)
- How much do these quantities change?... See board.

Answers:

\[
C_f = \frac{d}{d_f} C \\
E_f = \frac{d}{d_f} E \\
U_f = \frac{d}{d_f} U
\]

Where is the Energy stored?

- Claim: energy is stored in the electric field itself.
- Consider the example of a constant field generated by a parallel plate capacitor:

\[
U = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} \frac{Q^2}{(4d^2/C)}
\]

- The electric field is given by:

\[
E = \frac{Q}{\varepsilon_0 A} \quad \Rightarrow \quad U = \frac{1}{2} \varepsilon_0 E^2 Ad
\]

- The energy density \( u \) in the field is given by:

\[
U = \frac{u}{\text{volume}} = \frac{1}{2} \frac{\varepsilon_0 E^2}{d} = \frac{u}{\text{volume}} = \frac{1}{2} \varepsilon_0 E^2 Ad
\]

Units: \( \text{J/m}^2 \)