Today’s Topics

- Calculate Electric Field With Superposition (Direct Sum/Integral of Coulomb’s Law)
- Calculate Electric Field With Gauss’s Law
  - Gauss’s Law
  - Examples

Expected from preview:
Calculate E with continuous charge.
Surface, closed surface, surface integral, flux, the Gauss’s Law.
Review: Electric Field and Electric Force

Electric Field is a form of matter. It carries energy (later in the semester)
How to Calculate Electric Field?

- **Single point-like:**
  \[ \vec{E} = k_e \frac{q_0}{r^2} \hat{r} \]

- **Multiple charges:**
  \[ \vec{E} = k_e \sum \frac{q_i}{r_i^2} \hat{r}_i \]
  (superposition principle)

- **Continuous Charge Distribution:**
  \[ \vec{E} = k_e \lim_{\Delta q \to 0} \sum \frac{\Delta q_i}{r_i^2} \hat{r}_i = k_e \int \frac{dq}{r^2} \hat{r} \]

**Note:** For now, we assume charges are not moving. (electrostatic)
Example: Charged Rod

- A uniformly charged rod of length \( L \) has a total charge \( Q \), find the electric field:
  - at point A \( \rightarrow \) answer: \( E_x = -k_e Q/(a(L+a)) \), \( E_y = 0 \) (see board)
  - at point B \( \rightarrow \) answer: \( E_y = 2k_e Q/(Lb) \sin \theta_0 \), \( E_x = 0 \) \( \tan \theta_0 = -L/b \) (see board, show method only)
  - at an arbitrary point C. (see board, conceptual only).

Calculus Requirements in this Course

The level of calculus shown in this example:

1: Shall be understood at conceptual level.
2: Will be practiced in HW problems
3: Will not be tested in the exam.
(some less complicated forms might be in exams)
Example: Uniformly Charged Sphere

- A uniformly charged sphere has a radius \( a \) and total charge \( Q \), find the electric field outside and inside the sphere.

- Solution:

... 

Don’t take notes of my solution: I AM FOOLING AROUND!
It is very complicated with the superposition method!

⇒ Gauss’s Law to the rescue!
Electric Flux

- The electric flux through a surface element is defined as the dot product of the electric field and the surface area vector:
  \[ \Delta \Phi_E = \mathbf{E} \cdot \Delta \mathbf{A} = E \Delta A \cos \theta \]

- The net electric flux through a closed surface

\[ \Phi_E \equiv \oint \mathbf{E} \cdot d\mathbf{A} \]
Gauss’s Law

**Net electric flux** through any **closed** surface (“Gaussian surface”) equals the total charge enclosed inside the closed surface divided by the permittivity of free space.

\[ \Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{encl}}{\varepsilon_0} \]

- **electric flux** \( q_{encl} \): all **charges** enclosed regardless of positions
- \( \Phi_E \): net electric flux
- \( \oint \): line integral
- \( \mathbf{E} \): electric field
- \( d\mathbf{A} \): differential area
- \( \varepsilon_0 \): permittivity constant

\[ (4\pi\varepsilon_0)^{-1} = k \]
Trivia Quiz 1

- Compare electric fluxes through closed surfaces $s_1, s_2, s_3$:
  1. $\Phi_{s_1} > \Phi_{s_2} > \Phi_{s_3}$
  2. $\Phi_{s_1} = \Phi_{s_2} = \Phi_{s_3}$
  3. $\Phi_{s_1} < \Phi_{s_2} < \Phi_{s_3}$
Trivia Quiz 2

What is the electric flux through closed surface S?

1. $\Phi = 0$
2. $\Phi = \frac{q_1+q_2+q_3+q_4+q_5}{\varepsilon_0}$
3. $\Phi = \frac{q_1+q_2+q_3}{\varepsilon_0}$
Uniformly Charged Sphere Again

Solution using Gauss’s Law:

\[ \oint \mathbf{E} \cdot d\mathbf{A} = \frac{1}{\varepsilon_0} q_{in} \]

The setting is highly symmetrical:

→ Gaussian surface will be concentric sphere of radius \( r \).

How to evaluate \( \oint \mathbf{E} \cdot d\mathbf{A} \)?

Note the symmetry:

→ Direction of \( \mathbf{E} \): Radial
→ Magnitude of \( \mathbf{E} \): Same in all direction

\[ \oint \mathbf{E} \cdot d\mathbf{A} = \oint EdA = E \oint dA = EA = 4\pi r^2 E \]
Uniformly Charged Sphere: Details

\[ \oint \vec{E} \cdot d\vec{A} = 4\pi r^2 E = \frac{1}{\varepsilon_0} q_{in} \]

where: 
\[ q_{in} = \begin{cases} 
Q & \text{if } r>a \\
\frac{Q}{r^3} & \text{if } r<a 
\end{cases} \]

\[ E = \frac{1}{4\pi r^2 \varepsilon_0} q_{in} \]
Uniform Charge Sphere: Final Solution

Note:
This has the same form as the point charge

\[ E = \frac{k_e Q}{a^3} \]

inside outside
Procedure to Use Gauss’s Law

![Gauss's Law Equation]

- **General principle:**
  Gauss’s law is valid for any charge distributions, but practically it is useful only in limited situations where the charge distribution is highly symmetric.

- **Procedure:**
  1. Draw a Gaussian surface passing the field point concerned. Observe symmetry so that the surface integral is trivial.
     - **Direction of E:** Either perpendicular or parallel to the surface.
     - **Magnitude of E:** The same (or be zero) on the surfaces.
  2. Evaluate the surface integral using **arguments of symmetry**. And Equate the surface integral to \( \frac{q_{in}}{\varepsilon_0} \) and solve for \( E \).
Three Common Symmetric Cases

Spherical
(point Q, uniform sphere, shell)

Cylindrical
(infinite line/cylinder of Q)

Planar
(infinite sheet of Q)

The above symmetric settings give very predictable E

→ Direction: Normal to surfaces of same symmetry
→ Magnitude: Same across surface (of same symmetry)

\[ \oint \vec{E} \cdot d\vec{A} = E \, A \]
Another Example: Thin Spherical Shell

- Find the E field inside/outside a uniformly charged thin sphere.

Solution: Exercise with your TAs.

![Gaussian Surface for E(outside) and E(inside)]

Result: \(E_{\text{in}} = 0\)
### Table 24.1

**Typical Electric Field Calculations Using Gauss’s Law**

<table>
<thead>
<tr>
<th>Charge Distribution</th>
<th>Electric Field</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>Insulating sphere of radius $R$, uniform charge density, and total charge $Q$</td>
<td>$k_e \frac{Q}{r^2}$</td>
<td>$r &gt; R$</td>
</tr>
<tr>
<td></td>
<td>$k_e \frac{Q}{R^2} \cdot r$</td>
<td>$r &lt; R$</td>
</tr>
<tr>
<td>Thin spherical shell of radius $R$ and total charge $Q$</td>
<td>$k_e \frac{Q}{r^2}$</td>
<td>$r &gt; R$</td>
</tr>
<tr>
<td></td>
<td>$0$</td>
<td>$r &lt; R$</td>
</tr>
<tr>
<td>Line charge of infinite length and charge per unit length $\lambda$</td>
<td>$2k_e \frac{\lambda}{r}$</td>
<td>Outside the line</td>
</tr>
<tr>
<td>Infinite charged plane having surface charge density $\sigma$</td>
<td>$\frac{\sigma}{2\varepsilon_0}$</td>
<td>Everywhere outside the plane</td>
</tr>
<tr>
<td>Conductor having surface charge density $\sigma$</td>
<td>$\begin{cases} \frac{\sigma}{\varepsilon_0} \ 0 \end{cases}$</td>
<td>Just outside the conductor, Inside the conductor</td>
</tr>
</tbody>
</table>

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One More Exercise On Gauss’s Law

Two charges +2Q and –Q are placed at locations shown. Find the electric field at point P.

Solution:
1. Draw a Gaussian surface passing P
2. Apply Gauss’s law:
   \[ \oint E \cdot dA = \frac{q_{in}}{\varepsilon_0} \]
3. \( q_{in} = +2Q + (-Q) = Q \)
4. Surface integral:
   \[ \oint E \cdot dA = 4\pi r^2 E \]
5. \( \Rightarrow E = \frac{1}{4\pi \varepsilon_0} \left( \frac{Q}{r^2} \right) \)

Is this correct? No! Which step is wrong? step 4