NAME: SOLUTIONS

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- No books or notes are permitted. Use only the formula sheet provided with the exam.
- Write your final answer in the box provided.
- All answers should include units.
- To get credit for a problem you need to show your work in the space provided. If no work is shown you will get no credit, even if the answer in the box is correct. You are expected to work all problems using the basic laws of physics and the equations provided on the formula sheet. If you happen to remember the answer to a particular problem or know a shortcut formula you must still work the problem to get full credit.
- If you need more space, use the back of one of the sheets, and make a note that the work is continued on the back.
1) Particles are incident on the surface of a regular rectangular crystal in which the spacing between successive atomic layers is $a = 0.23 \text{ nm}$. Particles that scatter from the crystal are detected at the mirror angle, as shown. Find the lowest (non-zero) energy for which the scattering intensity is maximized due to constructive interference if the particles are:

a) electrons;

We need the pathlength difference to be $\lambda$.

From the picture the extra path length is

$$\Delta l = 2a \cos 30^\circ = 0.398 \text{ nm}$$

So $\lambda = 0.398 \text{ nm}$. Then $\lambda = \frac{h}{p}$ gives

$$p = \frac{h}{\lambda} \Rightarrow pc = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{0.398 \text{ nm}} = 3113 \text{ eV}$$

$$E = \frac{p^2}{2m} = \frac{(pc)^2}{2mc^2} = \frac{(3113 \text{ eV})^2}{2 \cdot 5.11 \times 10^8 \text{ eV}} = 9.48 \text{ eV}$$

b) x-rays.

x-rays are photons $\Rightarrow m = 0$, $E = pc \Rightarrow E = 3113 \text{ eV}$
2) If the world had only 2 dimensions instead of 3, the speed distribution for the atoms in a gas would be

\[ g(v) = C v e^{-mv^2/2kT}. \]

Calculate the average translational kinetic energy of these atoms. The following integrals may be useful: \( \int_0^\infty x^{2n+1} e^{-x^2} \, dx = \frac{1}{2} n! \).

**Normalize:**

\[ \int_0^\infty g(v) \, dv = 1 = C \int v e^{-mv^2/2kT} \, dv. \]

Let \( x^2 = \frac{mv^2}{2kT} \) \( \Rightarrow \) \( x = \sqrt{\frac{m}{2kT}} \, v \)

\[ dx = \frac{m}{2kT} \, dv \]

\[ 1 = C \left( \frac{2kT}{m} \right) \int_0^\infty x e^{-x^2} \, dx = C \left( \frac{2kT}{m} \right) \frac{1}{2} \quad \Rightarrow \quad C = \frac{m}{kT} \]

\[ E = \frac{1}{2} mv^2 \quad \text{so} \]

\[ \bar{E} = \int_0^\infty x \frac{1}{2} mv^2 g(v) \, dv = \frac{m}{2} \left( \frac{m}{kT} \right) \int_0^\infty v^3 e^{-mv^2/2kT} \, dv \]

Use the same substitution \( x^2 = \frac{mv^2}{2kT} \). Then

\[ \bar{E} = \frac{m}{2} \left( \frac{m}{kT} \right) \left( \sqrt{\frac{2kT}{m}} \right)^4 \int_0^\infty x^3 e^{-x^2} \, dx \]

\[ = \frac{m}{2} \left( \frac{m}{kT} \right) \left( \frac{2kT}{m} \right)^2 \frac{1}{2} = kT \]

In 2 dimensions there would be 2 degrees of freedom, so we should expect \( \bar{E} = 2 \left( \frac{1}{2} kT \right) \) \( \checkmark \)
3) An electron is trapped in a one-dimensional potential well

\[ V(x) = \begin{cases} \frac{-k}{x} & \text{for } x > 0 \\ \infty & \text{for } x < 0. \end{cases} \]

(a) Show that the function \( \psi(x) = Cx e^{-\alpha x} \) is a solution to the time-independent Schrödinger equation for the appropriate value of \( \alpha \). Calculate the energy of the electron for \( k = 2 \text{eV} \cdot \text{nm} \) and put your result in the answer box.

\[
\psi(x) = Cx e^{-\alpha x} \quad \Rightarrow \quad \frac{d}{dx} \psi = C \left[ e^{-\alpha x} - \alpha x e^{-\alpha x} \right] = C(1-\alpha x)e^{-\alpha x}
\]

\[
\frac{d^2}{dx^2} \psi = C \left[ (1-\alpha x)(-\alpha)e^{-\alpha x} - \alpha e^{-\alpha x} \right] = C \left[ -\alpha x + \alpha^2 x - \alpha \right] e^{-\alpha x}
\]

\[
= (\alpha^2 - \frac{\alpha}{x}) C x e^{-\alpha x}. \quad \text{So we need.}
\]

\[
E = -\frac{\hbar^2 \alpha^2}{2m}
\]

\[
-\frac{\hbar^2}{2m} \left( \alpha^2 - \frac{\alpha}{x} \right) C x e^{-\alpha x} - \frac{k}{x} C x e^{-\alpha x} = E C x e^{-\alpha x}
\]

\[
E = -\frac{1}{2} \left( \frac{2eV \cdot \text{nm}}{(197.3 \text{eV} \cdot \text{nm})^2} \right) \cdot \frac{k \text{me}^2}{(h \text{eV} \cdot \text{nm})^2} = -26.25 \text{ eV}
\]

(b) The electron in this quantum state has some probability \( P \) of being found in the classically forbidden region. Find the regions of \( x \) that are classically forbidden and show how you would calculate \( P \). You should write down the necessary integrals, but you do not need to evaluate them. The classical turning point is the point where

\[
E = V(x) \Rightarrow \quad -\frac{k^2 m}{2} \frac{1}{h^2} = -\frac{k}{x} \quad \Rightarrow \quad x = 2 \frac{h^2}{km} \equiv x_0
\]

\[
P = \int_{x_0}^{\infty} |\psi|^2 \, dx = |C|^2 \int_{x_0}^{\infty} x^2 e^{-2\alpha x} \, dx
\]

Normalization:

\[
1 = |C|^2 \int_{0}^{\infty} |\psi|^2 \, dx \Rightarrow |C|^2 = \frac{1}{\int_{0}^{\infty} x^2 e^{-2\alpha x} \, dx}
\]

\[
P = \frac{\int_{x_0}^{\infty} x^2 e^{-2\alpha x} \, dx}{\int_{0}^{\infty} x^2 e^{-2\alpha x} \, dx}
\]
4) A particle of mass $m$ is confined in the potential well shown in the drawing. Make careful sketches of the wave functions of the ground state and first excited state for this potential. In each case assume that the energy $E$ of the particle is less than $V_0$. You are encouraged to add comments and/or equations to further explain what you know about the mathematical form of $\psi(x)$ in each case.

**Comments**

**Ground State:**
- 0 nodes, symmetric about $x=0$.
- Exponential outside the wells + sinusoidal inside.

**1st Excited State:**
- 1 node, right at the center