Goals:
• Chapter 11 (Work)
  ✚ Employ conservative and non-conservative forces
  ✚ Relate force to potential energy
  ✚ Use the concept of power (i.e., energy per time)
• Chapter 12
  ✚ Define rotational inertia ("mass")
  ✚ Define rotational kinetic energy
  ✚ Define center of mass

Assignment:
• HW7 due Tuesday, Nov. 2nd
• For Wednesday: Read Chapter 12, Sections 1-3, 5 & 6

Exam 2
7:15 PM Thursday, Nov. 4th

Definition of Work, The basics

**Ingredients:** Force ($F$), displacement ($\Delta r$)

Work, $W$, of a constant force $F$ acts through a displacement $\Delta r$:

$$W = F \cdot \Delta r$$  
(Work is a scalar)

"Scalar or Dot Product" $F \cdot d\vec{r}$

If we know the angle the force makes with the path, the dot product gives us $F \cos \theta$ and $\Delta r$

$$\Delta W = \vec{F} \cdot \Delta \vec{r} = \left| \vec{F} \right| \left| \Delta \vec{r} \right| \cos \theta$$
**Exercise**
Work in the presence of friction and non-contact forces

- A box is pulled up a rough (μ > 0) incline by a rope-pulley-weight arrangement as shown below.
  - How many forces (including non-contact ones) are doing work on the box?
  - Of these which are positive and which are negative?
  - State the system (here, just the box)
  - Use a Free Body Diagram
  - Compare force and path

A. 2
B. 3
C. 4
D. 5

Work with varying forces or curved paths

If the path is curved or if the force varies with position then at each point

\[
dW = \vec{F} \cdot d\vec{r}
\]

and

\[
W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}
\]
Work and Varying Forces (1D)

- Consider a varying force $F(x)$

\[
\text{Area} = F_x \Delta x
\]

$F$ is increasing

Here $W = F \cdot \Delta r$

becomes $dW = F \, dx$

\[
W = \int_{x_i}^{x_f} F(x) \, dx
\]

Example: Spring-Mass energy transfer

- How much will the spring compress ($u = x - x_{equilibrium}$) to bring the box to a stop (i.e., $v = 0$) if the object is moving initially at a constant velocity ($v_0$) on frictionless surface as shown below?

\[
W_{\text{box}} = \int_{u_i}^{u_f} F(u) \, du
\]

\[
W_{\text{box}} = \int_{u_i}^{u_f} -ku \, du
\]

\[
W_{\text{box}} = -\frac{1}{2} k u_f^2 \bigg|_{u_i}
\]

\[
W_{\text{box}} = -\frac{1}{2} k \Delta u^2 = \Delta K
\]

\[
\frac{1}{2} k \Delta u^2 = \frac{1}{2} m v_0^2 - \frac{1}{2} m v_f^2
\]
Relate work to potential energy

- Consider the ball moving up to height $h$
  (from time 1 to time 2)
- How does this relate to the potential energy?

Work done by the Earth's gravity on the ball:

$$ W = F \cdot \Delta x = -mg \, h $$

$$ \Delta U = U_f - U_i = mg \, h - mg \, 0 = mg \, h $$

$$ \Delta U = -W $$

This is a general result for all conservative forces
(also path independent)

Conservative Forces & Potential Energy

- For any conservative force $F$ we can define a potential energy function $U$ in the following way:

$$ W = \int F \cdot dr = -\Delta U $$

The work done by a conservative force is equal and opposite to the change in the potential energy function.
Conservative Forces and Potential Energy

- So we can also describe work and changes in potential energy (for conservative forces)
  \[ \Delta U = - W \]
- but in 1D
  \[ W = F_x \Delta x \]
- Combining these two,
  \[ \Delta U = - F_x \Delta x \]
- Letting small quantities go to infinitesimals,
  \[ dU = - F_x dx \]
- Or,
  \[ F_x = -\frac{dU}{dx} \]

A Non-Conservative Force, Friction

- Looking down on an air-hockey table with no air flowing \((\mu > 0)\).
- Now compare two paths in which the puck starts out with the same speed \((K_{i\text{ path 1}} = K_{i\text{ path 2}})\).
A Non-Conservative Force

Since path_2 distance > path_1 distance the puck will be traveling slower at the end of path 2.

Work done by a non-conservative force irreversibly removes energy out of the “system”.

Here W_{NC} = E_{\text{final}} - E_{\text{initial}} < 0 \Rightarrow \text{and reflects } E_{\text{thermal}}

Work & Power:

● Two cars go up a hill, a Corvette and a ordinary Chevy Malibu. Both cars have the same mass.
● Assuming identical friction, both engines do the same amount of work to get up the hill.
● Are the cars essentially the same?
● NO. The Corvette can get up the hill quicker
● It has a more powerful engine.
Work & Power:

- Power is the rate at which work is done.

\[
\bar{P} = \frac{W}{\Delta t} \quad \quad P = \frac{dW}{dt}
\]

Units (SI) are Watts (W):

\[1 \text{ W} = 1 \text{ J} / \text{ s}\]

Example:

- A person of mass 200. kg walks up 10.0 m. If he/she climbs in 10.00 sec what is the average power used (g = 10 m/s^2)
- \[P_{\text{avg}} = \frac{F \cdot h}{t} = \frac{mgh}{t} = 200. \times 10.0 \times 10.0 / 10.00 \quad \text{Watts}\]
- \[P = 2000 \text{ W}\]

Work & Power:

- Average Power: \[\bar{P} = P_{\text{average}} = \frac{W}{\Delta t}\]
- Instantaneous Power: \[P = \frac{dW}{dt}\]
- Constant force \[P = \frac{dW}{dt} = \frac{d(F \cdot \ddot{x})}{dt} = F \cdot \ddot{x} = \vec{F} \cdot \vec{v}\]
Work & Power:

Example 2:

Engine of a jet develops a thrust of 15,000 N when plane is flying at 300 m/s. What is the horsepower of the engine?

\[ P = \vec{F} \cdot \vec{v} \]

\[ P = F \cdot v \]
\[ P = (15,000 \text{ N}) (300 \text{ m/s}) = 4.5 \times 10^6 \text{ W} \]
\[ = (4.5 \times 10^6 \text{ W}) (1 \text{ hp} / 746 \text{ W}) \sim 6,000 \text{ hp}! \]

Exercise

Work & Power

- Starting from rest, a car drives up a hill at constant acceleration and then suddenly stops at the top.
- The instantaneous power delivered by the engine during this drive looks like which of the following.

\[ P = \frac{dW}{dt} = \frac{mg\Delta h}{\Delta t} \]

A. Top
B. Middle
C. Bottom
Chap. 12: Rotational Dynamics

• Up until now rotation has been only in terms of circular motion with \( a_c = \frac{v^2}{R} \) and \( |a_T| = |\frac{d|v|}{dt}| \)

• Rotation is common in the world around us.

• Many ideas developed for translational motion are transferable.

Rotational Dynamics: A child’s toy, a physics playground or a student’s nightmare

• A merry-go-round is spinning and we run and jump on it. What does it do?

• We are standing on the rim and our “friends” spin it faster. What happens to us?

• We are standing on the rim a walk towards the center. Does anything change?
System of Particles (Distributed Mass):

- Until now, we have considered the behavior of very simple systems (one or two masses).
- But real objects have distributed mass!
- For example, consider a simple rotating disk and 2 equal mass $m$ plugs at distances $r$ and $2r$.

![Diagram of a rotating disk with two plugs at different distances]

- Compare the velocities and kinetic energies at these two points.

\[
1 \quad K = \frac{1}{2} m v^2 = \frac{1}{2} m (\omega r)^2
\]

- The rotation axis matters too!

\[
2 \quad K = \frac{1}{2} m (2v)^2 = \frac{1}{2} m (\omega 2r)^2
\]

- Twice the radius, four times the kinetic energy

\[
K = \frac{1}{2} m v^2 = \frac{1}{2} m (\omega r)^2
\]
Rotation & Kinetic Energy

- Consider the simple rotating system shown below. (Assume the masses are attached to the rotation axis by massless rigid rods).

- The kinetic energy of this system will be the sum of the kinetic energy of each piece:

\[ K = \frac{1}{2} \sum_{i=1}^{4} m_i v_i^2 \]

\[ K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_3 v_3^2 + \frac{1}{2} m_4 v_4^2 \]

- Notice that \( v_1 = \omega r_1 \), \( v_2 = \omega r_2 \), \( v_3 = \omega r_3 \), \( v_4 = \omega r_4 \)

- So we can rewrite the summation:

\[ K = \frac{1}{2} \sum_{i=1}^{4} m_i \omega^2 r_i^2 = \frac{1}{2} \sum_{i=1}^{4} m_i \omega^2 r_i^2 = \frac{1}{2} \left[ \sum_{i=1}^{4} m_i r_i^2 \right] \omega^2 \]

- We recognize the quantity, moment of inertia or \( I \), and write:

\[ K = \frac{1}{2} I \omega^2 \]
Calculating Moment of Inertia...

- For a single object, $I$ **depends** on the rotation axis!
- Example: $I_1 = 4 \text{ m} R^2 = 4 \text{ m} (2^{1/2} \text{ L} / 2)^2$

\[
I_1 = 2mL^2 \quad I_2 = mL^2 \quad I = 2mL^2
\]

Calculating Moment of Inertia...

- For a discrete collection of point masses we found:

\[
I = \sum_{i=1}^{N} m_i r_i^2
\]

- For a continuous solid object we have to add up the $mr^2$ contribution for every infinitesimal mass element $dm$.

\[
I = \int r^2 dm
\]
Moments of Inertia

- Some examples of I for solid objects:

Solid disk or cylinder of mass $M$ and radius $R$, about perpendicular axis through its center.

$$ I = \frac{1}{2} M R^2 $$

Lecture 16, Work & Energy

- Strings are wrapped around the circumference of two solid disks and pulled with identical forces for the same linear distance. Disk 1 has a bigger radius, but both are identical material (i.e. their density $\rho = \frac{M}{V}$ is the same). Both disks rotate freely around axes through their centers, and start at rest.

Which disk has the biggest angular velocity after the pull?

$$ W = F d = \frac{1}{2} I \omega^2 $$

(A) Disk 1  
(B) Disk 2  
(C) Same
Rotation & Kinetic Energy...

- The kinetic energy of a rotating system looks similar to that of a point particle:

<table>
<thead>
<tr>
<th>Point Particle</th>
<th>Rotating System</th>
</tr>
</thead>
</table>
| \[ K = \frac{1}{2} m v^2 \]  
  \( v \) is “linear” velocity  
  \( m \) is the mass. | \[ K = \frac{1}{2} I \omega^2 \]  
  \( \omega \) is angular velocity  
  \( I \) is the moment of inertia about the rotation axis.  
  \[ I = \sum_i m_i r_i^2 \] |

Connection with CM motion

- If an object of mass \( M \) is moving linearly at velocity \( V_{CM} \) without rotating then its kinetic energy is

\[ K_T = \frac{1}{2} M V_{CM}^2 \]

- If an object of moment of inertia \( I_{CM} \) is rotating in place about its center of mass at angular velocity \( \omega \) then its kinetic energy is

\[ K_R = \frac{1}{2} I_{CM} \omega^2 \]

- What if the object is both moving linearly and rotating?

\[ K = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M V_{CM}^2 \]
Connection with motion...

- So for a solid object which rotates about its center of mass and whose CM is moving:

\[
K_{\text{TOTAL}} = K_{\text{Rotational}} + \frac{1}{2} MV_{\text{CM}}^2
\]

System of Particles: Center of Mass (CM)

- If an object is not held then it will rotate about the center of mass.
- Center of mass: Where the system is balanced!
  - Building a mobile is an exercise in finding centers of mass.

\[ m_1 \quad \quad m_2 \quad \quad m_1 \quad \quad m_2 \]

mobile
System of Particles: Center of Mass

- How do we describe the “position” of a system made up of many parts?
- Define the **Center of Mass** (average position):
  - For a collection of $N$ individual point-like particles whose masses and positions we know:

$$\vec{R}_{\text{CM}} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{M}$$

(In this case, $N = 2$)

Sample calculation:

- Consider the following mass distribution:

$$\vec{R}_{\text{CM}} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{M} = X_{\text{CM}} \hat{i} + Y_{\text{CM}} \hat{j} + Z_{\text{CM}} \hat{k}$$

$X_{\text{CM}} = (m \times 0 + 2m \times 12 + m \times 24) / 4m$ meters

$Y_{\text{CM}} = (m \times 0 + 2m \times 12 + m \times 0) / 4m$ meters

$X_{\text{CM}} = 12$ meters

$Y_{\text{CM}} = 6$ meters

$R_{\text{CM}} = (12, 6)$
System of Particles: Center of Mass

- For a continuous solid, convert sums to an integral.

\[ \vec{R}_{CM} = \frac{\int \vec{r} \, dm}{\int dm} = \frac{\int \vec{r} \, dm}{M} \]

where \( dm \) is an infinitesimal mass element.

Recap

Assignment:
- HW7 due Nov. 2nd
- For Monday: Read Chapter 12, Sections 7-11