Ph 772 - Problem #5 - Solution

\[ F_x = 10^{-7} \text{ ergs/cm}^2/\text{s} \ (1-10 \text{ A}) \]

\[ R = 200 \text{ pc} \times 3 \times 10^{18} \text{ cm/pc} = 6 \times 10^{20} \text{ cm} \]

\[ \Theta = \frac{1}{2}'' \times \frac{10}{3600''} \times \frac{\pi \text{ rad}}{180^\circ} = \frac{1}{2} \times 4.85 \times 10^{-6} \text{ rad} = 2.4 \times 10^{-6} \text{ rad} \]

\[ V = \frac{4}{3} \pi r^3 = \frac{1}{6} \pi d^3 = \frac{1}{6} \pi \Theta^3 R^3 \]

\[ F_v = \frac{P_v V}{4\pi R^2} = \frac{1}{6} \pi \Theta^3 R^3 \]

\[ F_x = \int_{1A}^{10A} F_v \text{ d}v, \quad \int_{1A}^{10A} P_v \text{ d}v = \frac{24}{\Theta^3 R} F_x \]

Using (5.14.b) in R + L:

\[ P(v) = \frac{dW}{d\nu dt dv} = 6.8 \times 10^{-38} Z^2 \text{ne} \text{n}_i T^{-\frac{1}{2}} e^{-h\nu/kT} \frac{\text{ergs}}{\text{cm}^3 \text{ s} \text{ Hz}} \]

\[ V(1A) = \frac{C}{1A} = 3 \times 10^{18} \text{ s}^{-1} \quad V(10A) = 3 \times 10^{19} \text{ Hz} \]

\[ \int_{\nu}^{\nu_f} e^{-h\nu/kT} \text{ d}\nu = \frac{KT}{h} \left( e^{-h\nu/kT} - e^{-h\nu_f/kT} \right) = 1.04 \times 10^{18} \text{ s}^{-1} (0.75 - 0.06) \]

\[ = 7.2 \times 10^{12} \text{ Hz} \quad \text{(Note that this range contains almost all the energy)} \]

\[ \langle Z^2 \rangle \text{ for cosmic abundances } \sim 1.4 \text{ (0.9 from H, 0.4 from He, 0.1 from metals - mostly O + Ne)} \]

\[ \text{also } \text{n}_e \sim 1.1 \text{ ni } \Rightarrow \text{n}_e \text{n}_i \approx 0.9 \text{ n}_e^2 \]

\[ \frac{T}{h} \approx 0.7 \Rightarrow 1.4 \text{ from Fig 5.3 (1.2 \times 10^{-3}) [actually ranges from 0.7 - 1.5 over this } \nu \text{ range]} \]

\[ \Rightarrow F_v = 6.8 \times 10^{-38} (1.4)(0.9 \text{ n}_e^2)(5 \times 10^7 K)^{-\frac{1}{2}} (7.2 \times 10^{12} \text{ Hz})(1.4) \frac{\text{ergs}}{\text{cm}^3 \text{ s}} \]

\[ = 1.2 \times 10^{-23} \text{ n}_e^2 \text{ erg/cm}^3/\text{s} = \frac{24}{\Theta^3 R} F_x \]

\[ \Rightarrow \text{n}_e^2 = 2.3 \times 10^{13} / \text{cm}^6, \quad \text{Ne} = 4.8 \times 10^6 / \text{cm}^3 \text{ [} \propto (\Theta^3 R)^{-\frac{1}{2}} \text{]} \]
b) In radio, $h\nu < kT$, so use $R-T$ limit:

$$\chi_\nu = 0.18 \frac{1}{T^{3/2}} \pi^2 \text{Ne} \text{Ni} \nu^{-2} \sqrt{\text{ff}}$$

$$u = \frac{h\nu}{kT} \approx 3 \times 10^{-10} \text{Hz} \Rightarrow \Theta = 3 \times 10^{-9} \text{ at } 3000 \text{ MHz} \ (10 \text{ cm})$$

$$\therefore \chi_\nu \approx 8 \sqrt{\nu} \log^{-1}(\nu)$$

$$\chi_\nu \approx 5.1 \times 10^{-13} \text{Ne}^2 \nu^{-2} \text{cm}^{-1}$$

$$T_x \approx \chi_\nu \Theta \approx \chi_\nu \Theta R = 5.1 \times 10^{-13} \frac{\text{Ne}^2 \Theta R}{\nu^2} \left( \frac{F_x}{\Theta^2 \nu^2} \right)$$

$$= 1.7 \times 10^{16} / \nu^2 \Rightarrow \tau > 1 \text{ for } \nu < \sqrt{1.7 \times 10^{16}} = 130 \text{ MHz}$$

At frequencies less than this, the source will look like a black body of angular diameter $\Theta$, and the antenna temp. will be $= \theta$ source $T$ times the fraction of the beam filled by the source:

$$T_a \approx (\Theta/\Theta_0)^2 T_s$$

Where $\Theta_0 = \text{beamwidth} = \frac{1.22\lambda}{D} = \frac{1.22c}{D \nu}$

$$T_a \approx \left( \frac{\Theta}{\Theta_0} \right)^2 T_s \approx \left( \frac{\Theta}{1.22c} \right)^2 T_s = \frac{\left( \frac{\Theta}{1.22c} \right)^2 T_s}{\left( \frac{1.22c}{D \nu} \right)^2 T_s} = \frac{1.22c}{(4 \times 10^6 c)^2} (9.1 \times 10^{-5} \text{ cm/s})^2 = 5 \times 10^8 \text{ K}$$

$$= 0.6 \text{ K} \text{ at } 130 \text{ MHz} - \text{the highest frequency where this formula is valid. This is marginally undetectable according to the criterion in the problem, but at such a low frequency, } T_a \text{ will be dominated by galactic background, and the minimum detectable signal is much larger.}$$

Note that $T_a \propto \Theta^2 \nu^{-2}$, $\therefore \nu^2 \propto \frac{F_x}{\Theta^2}$, $\therefore \nu \propto F_x$, indep. of $R$ and $\Theta$. For $\nu > \nu_{\text{max}}$, $\tau < 1$ and $P(\nu)$ is indep. of $\nu$. Since $I_{B\nu} \propto \nu^2$, $T_{\text{eff}} = \left( \frac{\nu_{\text{max}}}{\nu} \right)^2 T_s$, and $T_a \propto \Theta^2 \nu^2$. $T_{\text{eff}} \propto \Theta^2$, and $T_a$ is indep. of $\nu$ for $\nu > \nu_{\text{max}}$.

We can check this. For radio frequencies where the source is optically thin, here $Fr = \frac{F_x}{\Delta x} \frac{g}{g_x}$

$$F_{\nu} \text{ (radio, optically thin)} = \frac{10^{-7} \text{ ergs/cm}^2/\text{Hz}}{7.2 \times 10^{11} \text{ Hz}} \frac{g}{g_x} = 8 \times 10^{-25} \text{ ergs/cm}^2/\text{s/Hz}$$

The amount of power collected by a 40 m antenna is then:

$$\frac{dU}{dt} = \pi \frac{D^2}{4} F_{\nu} = 1.8 \times 10^{-14} \text{ ergs/s/Hz}$$
Now this is the same $\frac{dW}{d\Omega d\nu}$ that would be collected if the beam were filled by a blackbody emitter at $T_A$:

$$\frac{dW}{d\Omega d\nu} = A \cdot \Omega \cdot I_{BB} = \frac{\pi}{4} D^2 \Theta_0^2 \frac{2\nu^2}{c^2} K_T A \left( \Theta_0 = \frac{1.22\lambda}{D} \right)$$

$$\approx 2.3 K T_A \text{ ergs/s/Hz} = 1.8 \times 10^{-18} \text{ ergs/s/Hz} \Rightarrow T_A = 0.04 K$$

(close enough, given approximations.) This is $T_A$ at all $\nu$ where $R - 3$ approx is valid and source is optically thin. (Note that $T_A \propto D^2$ for an unresolved source, so a slightly larger telescope could detect it.)

c) Optical: $F_{\nu} \text{ (optical)} = F_{\nu} \text{ (radio)} \frac{\text{Jy}}{\text{rad}} = F_{\nu} \text{ (radio)} \frac{5}{q}$

$$= 5 \times 10^{-25} \text{ ergs/cm}^2\text{s/Hz}$$

If you can find $M_V$ defined as an absolute energy flux weighted over the $V$-band ($\lambda = 5480 \text{ Å}$), you're all done. Less directly, given that an A0 star with $M_V = 0.7$ has $M_{bol} = 0$ and $T_e = 9700 K$, and $M_{bol} = 0$ is $2.52 \times 10^{-5} \text{ ergs/cm}^2\text{s}$:

$$\frac{4\pi R^2}{4\pi R^2} \frac{0 - T_e^4}{2.52 \times 10^{-5} \text{ ergs/cm}^2\text{s}}$$

by definition of $T_e$. Assuming the source spectrum to be a blackbody at $T = T_e$ (not too accurate, since column $T$ in visible is 15,400 K):

$$F_{\nu} = I_{BB} \cdot \Delta \nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \frac{\pi R^2}{R^2}$$

Using $\nu = \frac{c}{5480 \text{ Å}}$, $F_{5480 \text{ Å}} = 2.8 \times 10^{-20} \text{ ergs/cm}^2\text{s/Hz}$ For $M_V = 0.7$

$$\frac{5 \times 10^{-25}}{2.8 \times 10^{-20}} = 1.8 \times 10^{-5}$$

$$2.5 \log_{10} \left( 1.8 \times 10^{-5} \right) = 11.9 \text{ magnitudes} + 0.7$$

$\Rightarrow M_V \approx 12.6$ This is index of $R$ and $T$, until $T$ is so small that the source is optically thick @ 5000Å (unlikely).