1. a) Derive the Lorentz transformation for photon number density, \( \nu' \) photons cm\(^{-3} \), for photons all moving within some small solid angle (\( d\Omega' \) — not needed for this part of the problem) of direction \( \theta' \) as seen in \( K' \). Write the answer as \( \nu = F(\gamma, \beta, \mu) \nu' \), where \( \mu = \cos(\theta) \).

Hints:

#1. Have the observer in \( K' \) construct a physical box to enclose \( dV \), and count photon hits on the inside walls. Compare this operation as seen in \( K' \) and \( K \).

#2. The volume of the box is \( \gamma^{-1} \) smaller in the \( K \) system due to Lorentz contraction along the \( x \) axis. However, the answer is not \( \nu = \gamma \nu' \) (nor \( \nu = \gamma^{-1} \nu' \)) because the observer in \( K \) does not think the walls of the box are erected simultaneously.

#3. You only have to worry about the front and back walls (those perpendicular to the \( \beta \) vector, which is assumed to be along the \( x \) and \( x' \) axes): since the result must be independent of the shape of the box and any effects from the other sides can be made negligible by making the box arbitrarily short in \( x \) and large in \( y \) and \( z \), you know that effects at these other sides must exactly cancel.

#4. Have someone at the origin in \( K' \) send out a flash of light that arrives simultaneously at \( x' = \pm 1/2 \Delta x' \) to time the erection of the front and back walls. Decide how this looks in \( K \).

#5. The effect of the non-simultaneous erection of the walls depends on the propagation direction, \( \theta \). Get the sign on this right: \( \theta \) is the angle between the direction the photons are moving and the direction the \( K' \) system is moving relative to \( K \).

b) The only additional correction to get the differential energy density, \( \nu(\theta, \phi, \nu) \) ergs cm\(^{-3} \) sr\(^{-1} \) Hz (\( \nu' \nu = \nu'(h\nu') \) d\( \Omega r^{-1} \) d\( \nu r^{-1} \)), is to multiply by the ratio of solid angles observed in \( K' \) and \( K \) (since \( h\nu'/d\nu' = h\nu/d\nu \)). Rybicki & Lightman derive this ratio, if you need help.

c) Since \( I(\nu) \) ergs cm\(^{-2} \) s\(^{-1} \) sr\(^{-1} \) Hz\(^{-1} \) = \( r \nu \), you have now done the general Lorentz transformation of an intensity. Show that it is equivalent to the invariant: \( I(\nu)/\nu^3 = I'(\nu)/\nu'^3 \). (The latter is derived elegantly in Rybicki & Lightman.)